TASK Mathematics for Economic Applications 5/9/2016 $\boxed{\text{CMfEA1}}$

IM1) $(x-i)^2=\dfrac{1+i}{1-i} \Rightarrow (x-i)^2=\dfrac{(1+i)}{(1-i)}\cdot\dfrac{(1+i)}{(1+i)}=\dfrac{1+2i-1}{1+1}=\dfrac{2i}{2}=i \Rightarrow x-i=\sqrt{i}$.

$i=\cos\frac{\pi}{2}+i\,\mathrm{sen}\,\frac{\pi}{2} \Rightarrow \sqrt{i}=\cos\left(\frac{\pi}{4}+k\cdot\pi\right)+i\,\mathrm{sen}\left(\frac{\pi}{4}+k\pi\right),\ 0\le k\le 1$.

For $k=0$: $\sqrt{i}=\cos\frac{\pi}{4}+i\,\mathrm{sen}\,\frac{\pi}{4} \Rightarrow x=i+\cos\frac{\pi}{4}+i\,\mathrm{sen}\,\frac{\pi}{4}=\frac{\sqrt2}{2}+i\left(1+\frac{\sqrt2}{2}\right)$;

For $k=1$: $\sqrt{i}=\cos\frac{5}{4}\pi+i\,\mathrm{sen}\,\frac{5}{4}\pi \Rightarrow x=i+\cos\frac{5}{4}\pi+i\,\mathrm{sen}\,\frac{5}{4}\pi=-\frac{\sqrt2}{2}+i\left(1-\frac{\sqrt2}{2}\right)$.

IM2) From Rouchè-Capelli Theorem, to get ∞^3 solutions we need $\infty^{n-k}=\infty^{5-k} \Rightarrow 3 \Rightarrow$

$\Rightarrow$ RANK(A) = RANK(A|Y) = K = 2. Using elementary operations on the rows:

$$\left\|\begin{array}{ccccc|c}1&1&1&1&1&1\\1&-1&0&1&-1&1\\2&0&1&m&k&h\end{array}\right\| \xrightarrow[\substack{R_2\leftarrow R_2-R_1\\ R_3\leftarrow R_3-R_1-R_2}]{} \left\|\begin{array}{ccccc|c}1&1&1&1&1&1\\0&-2&-1&0&-2&0\\0&0&0&m-2&k&h-2\end{array}\right\|$$ and so, for $\begin{cases}m=2\\k=0\\h=2\end{cases} \Rightarrow$

$\Rightarrow$ RANK(A) = RANK(A|Y) = 2 $\Rightarrow \infty^{5-2}=\infty^3$ Solutions.

IM3) A set of three vectors is not a basis for $\mathbb{R}^3$ if the three vectors are linearly dependent vectors i.e. the determinant of the matrix having the three vectors as columns has a determinant equal to zero: $\begin{vmatrix}1&2&-1\\1&0&-1\\-2&3&k\end{vmatrix}=\begin{vmatrix}1&2&-1\\0&-2&0\\-2&3&k\end{vmatrix}=(-2)(k-2)=0$ iff $k=2$. For $k\neq2$ the three vectors form a basis for $\mathbb{R}^3$.

IM4) $A=\left\|\begin{matrix}1&0&1\\0&1&0\\1&0&1\end{matrix}\right\| \Rightarrow \begin{vmatrix}1-\lambda&0&1\\0&1-\lambda&0\\1&0&1-\lambda\end{vmatrix}=(1-\lambda)\left((1-\lambda)^2-1\right)=(1-\lambda)(\lambda^2-2\lambda)=\lambda\cdot(1-\lambda)\cdot(\lambda-2)=0$

for $\lambda_1=0$; $\lambda_2=1$; $\lambda_3=2$. Now we check for three eigenvectors.

$\|A-0\cdot\mathbb{I}\|\cdot X=\underline{0} \Rightarrow \left\|\begin{matrix}1&0&1\\0&1&0\\1&0&1\end{matrix}\right\|\cdot\left\|\begin{matrix}x\\y\\z\end{matrix}\right\|=\underline{0} \Rightarrow \begin{cases}x+z=0\\y=0\\x+z=0\end{cases} \Rightarrow \begin{cases}z=-x\\y=0\end{cases} \Rightarrow (1;0;-1) \Rightarrow \left(\frac{1}{\sqrt2};0;-\frac{1}{\sqrt2}\right)$ unit vector.

$\|A-1\cdot\mathbb{I}\|\cdot X=\underline{0} \Rightarrow \left\|\begin{matrix}0&0&1\\0&0&0\\1&0&0\end{matrix}\right\|\cdot\left\|\begin{matrix}x\\y\\z\end{matrix}\right\|=\underline{0} \Rightarrow \begin{cases}z=0\\\forall y\\x=0\end{cases} \Rightarrow (0;1;0)$ (unit vector)

$\|A-2\,\mathbb{I}\|\cdot X=\underline{0} \Rightarrow \left\|\begin{matrix}-1&0&1\\0&-1&0\\1&0&-1\end{matrix}\right\|\cdot\left\|\begin{matrix}x\\y\\z\end{matrix}\right\|=\underline{0} \Rightarrow \begin{cases}-x+z=0\\-y=0\\x-z=0\end{cases} \Rightarrow \begin{cases}z=x\\y=0\end{cases} \Rightarrow (1;0;1) \Rightarrow \left(\frac{1}{\sqrt2};0;\frac{1}{\sqrt2}\right)$ unit vector

Ortogonal Matrix: $U=\left\|\begin{matrix}\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\\0&1&0\\-\frac{1}{\sqrt2}&0&\frac{1}{\sqrt2}\end{matrix}\right\| \Rightarrow U^T\cdot A\cdot U=D=\left\|\begin{matrix}0&0&0\\0&1&0\\0&0&2\end{matrix}\right\|$.

II M1) $f(x;y) = x^2y - Kxy + xy^2$. $\nabla f(x;y) = (0;0) \Rightarrow$

$$\Rightarrow \begin{cases} f'_x = 2xy - Ky + y^2 = y \cdot (2x - K + y) = 0 \\ f'_y = x^2 - Kx + 2xy = x \cdot (x - K + 2y) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \cup \begin{cases} x = 0 \\ -K + y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = K \end{cases} \cup$$

$$\cup \begin{cases} x - K = 0 \\ y = 0 \end{cases} \Rightarrow \begin{cases} x = K \\ y = 0 \end{cases} \cup \begin{cases} y = K - 2x \\ x - K + 2K - 4x = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3}K \\ y = \frac{1}{3}K \end{cases}. \quad P_0 = (0;0);\ P_1 = (0;K);\ P_2 = (K;0);\ P_3 = \left(\frac{K}{3};\frac{K}{3}\right).$$

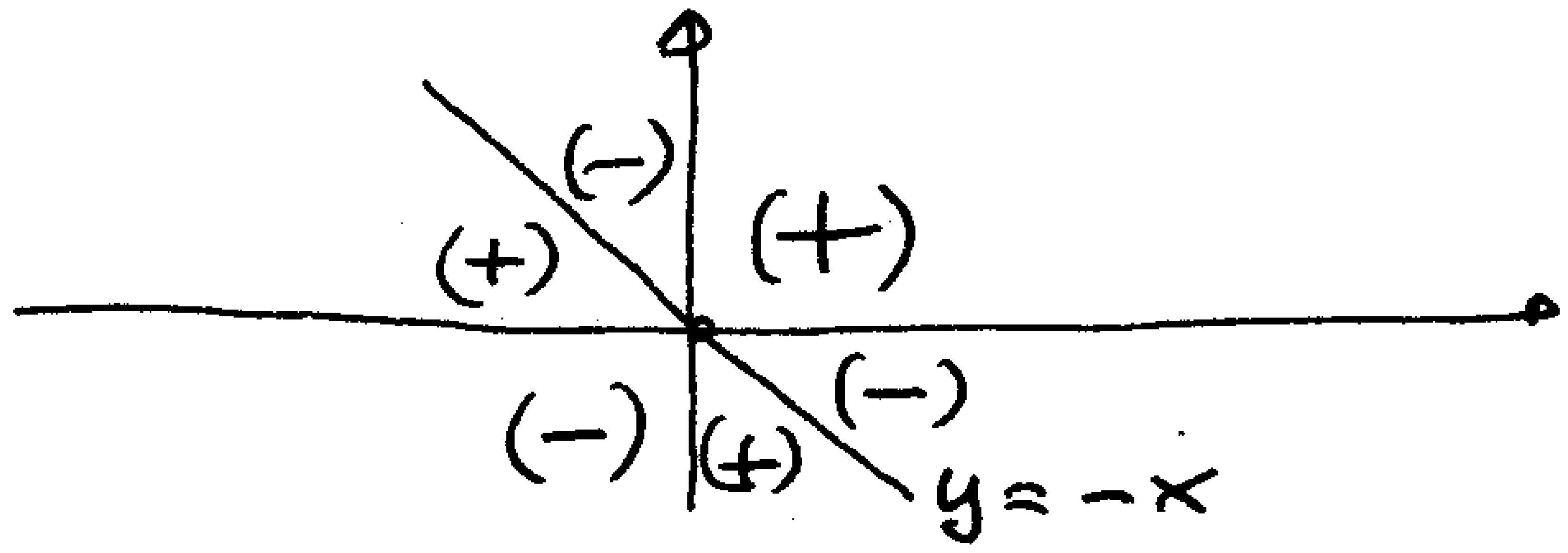
$$H(x;y) = \begin{Vmatrix} 2y & 2x - K + 2y \\ 2x - K + 2y & 2x \end{Vmatrix}. \quad H(0;0) = \begin{Vmatrix} 0 & -K \\ -K & 0 \end{Vmatrix} : \text{Saddle Point}; \quad H(0;K) = \begin{Vmatrix} 2K & K \\ K & 0 \end{Vmatrix} : \text{Saddle Point};$$

$$H(K;0) = \begin{Vmatrix} 0 & K \\ K & 2K \end{Vmatrix} : \text{Saddle Point}; \quad H\left(\tfrac{1}{3}K;\tfrac{1}{3}K\right) = \begin{Vmatrix} \frac{2}{3}K & \frac{1}{3}K \\ \frac{1}{3}K & \frac{2}{3}K \end{Vmatrix} \begin{cases} \text{minimum point for } K > 0 \\ \text{Maximum point for } K < 0 \end{cases}.$$

For $K = 0$: $f(x;y) = x^2y + xy^2 = xy(x+y)$. Only $(0;0)$ is a stationary point.

$H(0;0) = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}$. $f(0;0) = 0$; $f(x;y) > 0$:

$$x \cdot y \cdot (x+y) > 0 : \begin{cases} x > 0 \\ y > 0 \\ y > -x \end{cases}$$



So $(0;0)$ is a saddle point.

II M2) $f(x;y) = 2xy - xy^2$. $\nabla f(x;y) = (2y - y^2;\ 2x - 2xy) \Rightarrow \nabla f(1;-1) = (-3;4)$.

$D_v f(1;-1) = \nabla f(1;-1) \cdot (\cos\alpha; \operatorname{sen}\alpha) = (-3;4)(\cos\alpha;\operatorname{sen}\alpha) = -3\cos\alpha + 4\operatorname{sen}\alpha$.

We see that: $-3\cos\alpha + 4\operatorname{sen}\alpha = \frac{7}{\sqrt{2}}$ if $\cos\alpha = -\frac{1}{\sqrt{2}}$ and $\operatorname{sen}\alpha = \frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{3}{4}\pi$.

II M3) $f(x;y;z) = xyz - x^2y + xz^2 - yz = 0$. $f(1;1;1) = 1 - 1 + 1 - 1 = 0$.

$\nabla f(x;y;z) = (yz - 2xy + z^2;\ xz - x^2 - z;\ xy + 2xz - y)$. $\nabla f(1;1;1) = (0;-1;2)$.

It is possible to define: $(x;z) \to y \Rightarrow \frac{\partial y}{\partial x} = -\frac{0}{-1} = 0$ and $\frac{\partial y}{\partial z} = -\frac{2}{-1} = 2$; OR

It is possible to define: $(x;y) \to z \Rightarrow \frac{\partial z}{\partial x} = -\frac{0}{2} = 0$ and $\frac{\partial z}{\partial y} = -\frac{-1}{2} = \frac{1}{2}$.

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II M4) $\begin{cases} \text{Max/min } f(x;y) = x^2+y^2 \\ \text{u.c.}: 4x^2+y^2=4. \end{cases}$

f is a continuos function
Σ is a closed and limited set.
From Weierstrass Theorem Max and min exist.

$\Lambda = x^2+y^2-\lambda(4x^2+y^2-4)$

$\begin{cases} \Lambda'_x = 2x-8\lambda x = 2x(1-4\lambda)=0 \\ \Lambda'_y = 2y-2\lambda y = 2y(1-\lambda)=0 \\ 4x^2+y^2=4 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ 0+0=4: \text{No!} \end{cases} \cup \begin{cases} x=0 \\ \lambda=1 \\ y^2=4 \end{cases} \Rightarrow \begin{cases} x=0 \\ \lambda=1 \\ y=\pm 2 \end{cases} \cup \begin{cases} \lambda=\frac{1}{4} \\ y=0 \\ x^2=1 \end{cases} \Rightarrow \begin{cases} \lambda=\frac{1}{4} \\ x=\pm 1 \\ y=0 \end{cases}.$

There are 4 solutions: $P_1=(0;2)$; $P_2=(0;-2)$ with $\lambda=1$; $P_3=(1;0)$; $P_4=(-1;0)$ with $\lambda=\frac{1}{4}$.

$$\overline{H}(x;y;\lambda) = \left\|\begin{matrix} 0 & 8x & 2y \\ 8x & 2-8\lambda & 0 \\ 2y & 0 & 2-2\lambda \end{matrix}\right\|.$$

$$|\overline{H}(0;2;1)| = \begin{vmatrix} 0 & 0 & 4 \\ 0 & -6 & 0 \\ 4 & 0 & 0 \end{vmatrix} = 4\cdot\begin{vmatrix} 0 & 4 \\ -6 & 0 \end{vmatrix} = 4\cdot 24 > 0 : \text{Maximum point.}$$

$$|\overline{H}(0;-2;1)| = \begin{vmatrix} 0 & 0 & -4 \\ 0 & -6 & 0 \\ -4 & 0 & 0 \end{vmatrix} = (-4)\cdot\begin{vmatrix} 0 & -4 \\ -6 & 0 \end{vmatrix} = (-4)(-24) > 0 : \text{Maximum point.}$$

$$|\overline{H}(1;0;\tfrac{1}{4})| = \begin{vmatrix} 0 & 8 & 0 \\ 8 & 0 & 0 \\ 0 & 0 & \frac{3}{2} \end{vmatrix} = \frac{3}{2}\cdot\begin{vmatrix} 0 & 8 \\ 8 & 0 \end{vmatrix} = \frac{3}{2}\cdot(-64) < 0 : \text{Minimum point.}$$

$$|\overline{H}(-1;0;\tfrac{1}{4})| = \begin{vmatrix} 0 & -8 & 0 \\ -8 & 0 & 0 \\ 0 & 0 & \frac{3}{2} \end{vmatrix} = \frac{3}{2}\cdot\begin{vmatrix} 0 & -8 \\ -8 & 0 \end{vmatrix} = \frac{3}{2}\cdot(-64) < 0 : \text{Minimum point.}$$

$f(0;2) = f(0;-2) = 4$: Maximum value for f.

$f(1;0) = f(-1;0) = 1$: Minimum value for f.