Intermediate Test of Mathematics for Economic Applications 02/12/2016

I M 1) $z = (2i)^{19} \cdot (1+i)^{16} = 2^{19} \cdot i^{19} \cdot \left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right)^{16} =$
$2^{19} \cdot i^3 \cdot 2^8(\cos(4\pi) + i\sin(4\pi)) = -2^{27}i$ and $\overline{z} = 2^{27}i$; $\sqrt{\overline{z}} = \sqrt{2^{27}i} =$
$2^{\frac{27}{2}}\sqrt{\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)} = 2^{\frac{27}{2}}\left(\cos\left(\frac{\pi}{4} + k\pi\right) + i\sin\left(\frac{\pi}{4} + k\pi\right)\right)$ with $k = 0, 1$.
We conclude that $\sqrt{\overline{z}} = \pm 2^{\frac{27}{2}}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \pm 2^{13}(1+i)$.

I M 2) Since the system is an homogeneus one, it has ∞^1 solutions if and only if $n - \text{Rank}(\mathbb{C}) = 1$, where n is the number of the unknown variables and $\mathbb{C}$ is the matrix of the coefficients. To check the rank we reduce the matrix by elementary operations on its rows: $\left\|\begin{matrix} m & n & 1 & 1 \\ m & n & k & 1 \\ m & n & k & h \end{matrix}\right\| \overset{\substack{R_2 - R_1 \\ R_3 - R_2}}{\longrightarrow} \left\|\begin{matrix} m & n & 1 & 1 \\ 0 & 0 & k-1 & 0 \\ 0 & 0 & 0 & h-1 \end{matrix}\right\|$ and it is easy to note that $\text{Rank}(\mathbb{C}) = 3$ iff $h \neq 1$, $k \neq 1$ and at least one from m and n is different from 0.

I M 3) If $\mathbb{A}$ is the matrix associated to the linear map F, we know that $\dim(\text{Imm}(F))$ is equal to $\text{Rank}(\mathbb{A})$.
Like the previous exercise we can check $\text{Rank}(\mathbb{A})$ by elementary operations on the lines of $\mathbb{A}$: $\left\|\begin{matrix} m & 1 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 1 & m \end{matrix}\right\| \overset{\substack{R_1 - mR_2 \\ R_4 - mR_3}}{\longrightarrow} \left\|\begin{matrix} 0 & 1-mk & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 0 & k & 1 \\ 0 & 0 & 1-mk & 0 \end{matrix}\right\| \overset{\substack{R_1 \circlearrowleft R_2 \\ C_3 \circlearrowleft C_4}}{\longrightarrow}$

$\longrightarrow \left\|\begin{matrix} 1 & k & 0 & 0 \\ 0 & 1-mk & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1-mk \end{matrix}\right\|$ and so we have:

$\text{Rank}(\mathbb{A}) = \dim(\text{Imm}(F)) = \begin{cases} 4 & \text{if } mk \neq 1 \\ 2 & \text{if } mk = 1 \end{cases}.$

By Sylvester Theorem we know that $\dim(\text{Ker}) + \dim(\text{Imm}) = \dim(\mathbb{R}^4) = 4$; if $\dim(\text{Imm})$ is minimum ($mk = 1$) it trivialy follows $\dim(\text{Imm}) = \dim(\text{Ker}) = 2$.
To find a basis for the Kernel remember that $\mathbb{X}$ belongs to $\text{Ker}(F)$ if $\mathbb{A} \cdot \mathbb{X} = \mathbb{O}$, that in system form is: $\begin{cases} mx_1 + x_2 = 0 \\ x_1 + kx_2 = 0 \\ kx_3 + x_4 = 0 \\ x_3 + mx_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -kx_2 \\ x_4 = -kx_3 \end{cases}$. Choosing $k \neq 0$ and $m = 1/k$, every element of $\text{Ker}(F)$ is of the form:
$\mathbb{X} = (-kx_2, x_2, x_3, -kx_3) = x_2(-k, 1, 0, 0) + x_3(0, 0, 1, -k)$.
So a basis for $\text{Ker}(F)$ is $\mathcal{B}_{\text{Ker}(F)} = \{(-k, 1, 0, 0), (0, 0, 1, -k)\}$.
For a basis of $\text{Imm}(F)$ remember that $\mathbb{Y}$ belongs to the immage of F if $\mathbb{A} \cdot \mathbb{X} = \mathbb{Y}$, that in system form is:

$$\begin{cases} mx_1 + x_2 = y_1 \\ x_1 + kx_2 = y_2 \\ kx_3 + x_4 = y_3 \\ x_3 + mx_4 = y_4 \end{cases} \Rightarrow \begin{cases} (1/k)\,x_1 + x_2 = y_1 \\ x_1 + kx_2 = y_2 \\ kx_3 + x_4 = y_3 \\ x_3 + (1/k)\,x_4 = y_4 \end{cases} \Rightarrow \begin{cases} x_1 + kx_2 = ky_1 \\ x_1 + kx_2 = y_2 \\ kx_3 + x_4 = y_3 \\ kx_3 + x_4 = ky_4 \end{cases} \Rightarrow \begin{cases} y_2 = ky_1 \\ y_3 = ky_4 \end{cases}.$$

Choosing again $k \neq 0$ and $m = 1/k$, every element of Imm(F) is of the form:
$\mathbb{Y} = (y_1, y_2, y_3, y_4) = (y_1, ky_1, ky_4, y_4) = y_1(1, k, 0, 0) + y_4(0, 0, k, 1)$ and a basis for Imm(F) is $\mathcal{B}_{\text{Imm}(F)} = \{(1, k, 0, 0), (0, 0, k, 1)\}$.

I M 4) Since the matrix $\mathbb{A}_2$ has eigenvectors with two components, thus $\mathbb{A}_2$ is a 2×2 square matrix; if $\mathbb{A}_2 = \left\| \begin{matrix} x & y \\ z & w \end{matrix} \right\|$ by the conditions the matrix $\mathbb{A}_2$ must satisfy:

$\mathbb{A}_2 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbb{A}_2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, that in system form is:

$$\begin{cases} x + y = 1 \\ z + w = 1 \\ x - y = -1 \\ z - w = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ z = 1 \\ w = 0 \end{cases} \Rightarrow \mathbb{A}_2 = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|.$$

I M 5) The characteristic polynomial of $\mathbb{A}$ is:

$$p_{\mathbb{A}}(\lambda) = \begin{vmatrix} 2-\lambda & 1 & -1 \\ 1 & 2-\lambda & -1 \\ -1 & -1 & k-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 0 & 2-\lambda & -1 \\ k-\lambda-1 & -1 & k-\lambda \end{vmatrix} =$$

$= (1-\lambda)((2-\lambda)(k-\lambda) - 1) + (k-\lambda-1)(1-\lambda) =$

$= (1-\lambda)\big(\lambda^2 - (k+3)\lambda + 3k - 2\big) = 0$, so $\forall k$: $p_{\mathbb{A}}(1) = 0$, and $\lambda = 1$ is an eigenvalue of $\mathbb{A}$ for every k. Define $q_{\mathbb{A}}(\lambda) = -(\lambda^2 - (k+3)\lambda + 3k - 2)$.

If $\lambda = 1$ is a multiple eigenvalue of $\mathbb{A}$, $q_{\mathbb{A}}(1) = 4 - 2k = 0$ and so $k = 2$.

Since $\lambda^2 - 5\lambda + 4 = (\lambda - 1)(4 - \lambda)$, the characteristic polynomial is factorized as $p_{\mathbb{A}}(\lambda) = (\lambda - 1)^2(4 - \lambda)$ and the eigenvalues of $\mathbb{A}$ are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 4$.

Since $\mathbb{A}$ is a symmetric matrix, from the Spectral Theorem we know that $\mathbb{A}$ is a diagonalizable matrix and so $m_1^g = 2 = m_1^a$: the geometric multiplicity of $\lambda = 1$, the dimention of $\mathcal{ES}_{\lambda=1}$, the eigenspace associated to the eigenvalue $\lambda = 1$, is equal to 2, its algebraic multiplicity. Solving the homogeneus system $(\mathbb{A} - 1 \cdot \mathbb{I}) \cdot \mathbb{X} = \mathbb{O}$ we get:

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ x_1 + x_2 - x_3 = 0 \\ -x_1 - x_2 + x_3 = 0 \end{cases} \Rightarrow x_3 = x_1 + x_2.$$ Every element of $\mathcal{ES}_{\lambda=1}$ is a vector:

$\mathbb{X} = (x_1, x_2, x_1 + x_2)$ but for $\lambda_1 = \lambda_2 = 1$ we must find two orthogonal unit vectors.

For $x_1 = 1$ and $x_2 = 0$ we get $\mathbb{X}_1 = (1, 0, 1)$. To get another eigenvector orthogonal to $\mathbb{X}_1$ we pose: $(1, 0, 1) \cdot (x_1, x_2, x_1 + x_2) = 0$ and we get:

$x_1 + x_1 + x_2 = 0 \Rightarrow x_2 = -2x_1$ and so the eigenvector $\mathbb{X}_2 = (x_1, -2x_1, -x_1)$.

For $x_1 = 1$ we get $\mathbb{X}_2 = (1, -2, -1)$. From $\mathbb{X}_1$ and $\mathbb{X}_2$ we get the unit eigenvectors:

$\mathbb{V}_1 = \left(\sqrt{2}/2, 0, \sqrt{2}/2\right)$, $\mathbb{V}_2 = \left(\sqrt{6}/6, -\sqrt{6}/3, -\sqrt{6}/6\right)$.

For $\lambda_3 = 4$ the homogeneus system $(\mathbb{A} - 4 \cdot \mathbb{I}) \cdot \mathbb{X} = \mathbb{O} \Rightarrow$

$$\begin{cases} -2x_1 + x_2 - x_3 = 0 \\ x_1 - 2x_2 - x_3 = 0 \\ -x_1 - x_2 - 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -2x_1 + x_2 \\ 3x_1 - 3x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -x_1 \\ x_2 = x_1 \end{cases}.$$ Every element of $\mathcal{ES}_{\lambda=4}$ is a vector $\mathbb{X} = (x_1, x_1, -x_1)$. For $x_1 = 1$ we get $\mathbb{X}_3 = (1, 1, -1)$ and the unit eigenvector $\mathbb{V}_2 = \left(\sqrt{3}/3, \sqrt{3}/3, -\sqrt{3}/3\right)$.

The orthogonal requested matrix is $\mathbb{U} = \left\| \begin{matrix} \sqrt{2}/2 & \sqrt{6}/6 & \sqrt{3}/3 \\ 0 & -\sqrt{6}/3 & \sqrt{3}/3 \\ \sqrt{2}/2 & -\sqrt{6}/6 & -\sqrt{3}/3 \end{matrix} \right\|$.