Compito di Matematica Generale del 4/7/2014

MG1

1) $f(x) = e^{2x} - 2e^{x-1} = e^x\left(e^x - \frac{2}{e}\right)$. C.E.: $\mathbb{R}$. $\lim_{x\to-\infty} f(x) = 0^-$; $\lim_{x\to+\infty} f(x) = +\infty$.

$f(x) \geq 0$: $e^x - \frac{2}{e} \geq 0 \Rightarrow e^x \geq \frac{2}{e} \Rightarrow x \geq \log\frac{2}{e}$. $f(0) = 1 - \frac{2}{e}$

$f'(x) = 2e^{2x} - 2e^{x-1} = 2e^x\left(e^x - \frac{1}{e}\right) \geq 0 \Rightarrow e^x \geq \frac{1}{e} \Rightarrow x \geq \log\frac{1}{e} = -1$.

$f''(x) = 4e^{2x} - 2e^{x-1} = 2e^x\left(2e^x - \frac{1}{e}\right) \geq 0 \Rightarrow e^x \geq \frac{1}{2e} \Rightarrow x \geq \log\frac{1}{2e}$

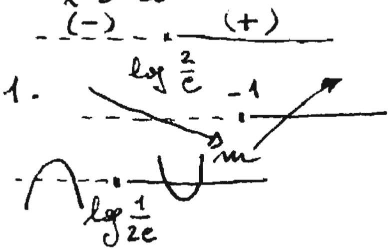
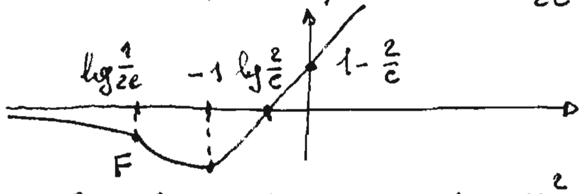


Grafico:



2) $\lim_{x\to0} \frac{1-\cos 2x}{\sin^2 x} = \lim_{x\to0} \frac{1-\cos 2x}{(2x)^2} \cdot 4 \cdot \frac{x^2}{\sin^2 x} = \frac{1}{2} \cdot 4 \cdot 1 = 2$.

$\lim_{x\to-\infty} \frac{e^{3x} - e^x + x^2}{e^{2x} + x^3} = \lim_{x\to-\infty} \frac{x^2}{x^3} = \lim_{x\to-\infty} \frac{1}{x} = 0^-$ $(e^{3x} \to 0;\ e^x \to 0;\ e^{2x} \to 0)$.

3) Equazione rette passanti per $(0;2)$: $y - 2 = m(x-0) \Rightarrow y = mx + 2$.

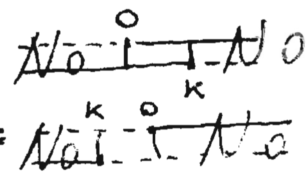
Mettendo a sistema $\begin{cases} y = mx+2 \\ y = 2x - x^2 \end{cases} \Rightarrow mx + 2 = 2x - x^2 \Rightarrow x^2 + (m-2)x + 2 = 0$.

Per avere tangenza: $\Delta = 0$: $(m-2)^2 - 4\cdot2\cdot1 = m^2 - 4m + 4 - 8 = m^2 - 4m - 4 = 0 \Rightarrow$

$\Rightarrow m = 2 \pm \sqrt{4+4} = 2 \pm \sqrt{8} = 2 \pm 2\sqrt{2}$. Ci sono due soluzioni:

$y = (2+2\sqrt{2})x + 2$ e $y = (2-2\sqrt{2})x + 2$.

Se $k>0$:
Se $k<0$:

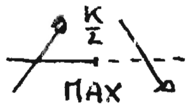


4) $f(x) = \log(kx - x^2)$. C.E: $kx - x^2 > 0 \Rightarrow x(k-x) > 0$

Se $k>0$: C.E. $=]0;k[$; Se $k<0$: C.E $=]k;0[$.

Se $k=0$: C.E. $= \emptyset$.

$f'(x) = \frac{k-2x}{kx-x^2} \geq 0 \Rightarrow k - 2x \geq 0 \Rightarrow x \leq \frac{k}{2}$



$\Rightarrow \frac{k}{2} = 3 \Rightarrow k = 6$.

5) $\int_1^e x\log x - \frac{1}{x}\,dx \Rightarrow \frac{x^2}{2}\cdot\log x - \int \frac{x^2}{2}\cdot\frac{1}{x}\,dx - \log x \Rightarrow \left(\frac{x^2}{2}\log x - \frac{x^2}{4} - \log x\right|_1^e =$

$= \left(\frac{e^2}{2}\log e - \frac{e^2}{4} - \log e\right) - \left(\frac{1}{2}\log 1 - \frac{1}{4} - \log 1\right) = \frac{e^2}{2} - \frac{e^2}{4} - 1 - 0 + \frac{1}{4} + 0 = \frac{e^2}{4} - \frac{3}{4} = \frac{1}{4}(e^2 - 3)$.

6) $A\cdot B\cdot X = \begin{Vmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 2 & 1 \\ 1 & 0 \\ 1 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} 2-1+2 & 1-0-2 \\ 2+0+1 & 1+0-1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} 3 & -1 \\ 3 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} 3x-y \\ 3x \end{Vmatrix}$

$A\cdot B\cdot X \parallel (1;2) \Rightarrow \frac{3x-y}{1} = \frac{3x}{2} \Rightarrow 6x - 2y = 3x \Rightarrow 3x = 2y \Rightarrow y = \frac{3}{2}x$.

$A\cdot B\cdot X \perp (1;2) \Rightarrow (3x-y;3x)\cdot(1;2) = 3x - y + 6x = 0 \Rightarrow 9x - y = 0 \Rightarrow y = 9x$.

7) $f(x) = x^3 - 3x^2 + 3$: polinomio, quindi continua e derivabile $\forall x \in \mathbb{R}$.

$f(0) = f(k) \Rightarrow 3 = k^3 - 3k^2 + 3 \Rightarrow k^3 - 3k^2 = k^2(k-3) = 0 \Rightarrow k = 3$. $(k \neq 0!!)$.

$f'(x) = 3x^2 - 6x = 3x(x-2) = 0 \Rightarrow x = 2 \in [0;3]$.

$x = 2$ è il punto stazionario del Teorema di Rolle.

MG2

8) $f(x;y) = x^2 \cdot \log(x-y+1) - xy\,e^y$.

C.E. : $x-y+1>0 \Rightarrow y < x+1 \Rightarrow (0;0) \in$ C.E.

$$\nabla f(x;y) = (f'_x ; f'_y) = \left(2x\log(x-y+1) + \frac{x^2}{x-y+1} - ye^y ; \frac{-x^2}{x-y+1} - x(1\cdot e^y + ye^y)\right).$$

$$\nabla f(0;0) = (0+0-0 ; 0-0\cdot 1) = (0;0).$$

9) $f(x) = 3x-1$; $g(x) = \log_2 x$; $h(x) = 2-3x$.

$$f(g(h(x))) = f(g(2-3x)) = f(\log_2(2-3x)) = 3\cdot\log_2(2-3x) - 1 = y \Rightarrow$$

$$\Rightarrow y+1 = 3\log_2(2-3x) \Rightarrow \frac{y+1}{3} = \log_2(2-3x) \Rightarrow 2-3x = 2^{\frac{y+1}{3}} \Rightarrow 3x = 2-2^{\frac{y+1}{3}} \Rightarrow$$

$$\Rightarrow x = \frac{1}{3}\left(2-2^{\frac{y+1}{3}}\right) \Rightarrow \text{Inversa}: y = \frac{1}{3}\left(2-2^{\frac{x+1}{3}}\right).$$

$$h(g(f(x))) = h(g(3x-1)) = h(\log_2(3x-1)) = 2-3\log_2(3x-1) = y \Rightarrow$$

$$\Rightarrow 3\log_2(3x-1) = 2-y \Rightarrow \log_2(3x-1) = \frac{2-y}{3} \Rightarrow 3x-1 = 2^{\frac{2-y}{3}} \Rightarrow 3x = 1+2^{\frac{2-y}{3}} \Rightarrow$$

$$\Rightarrow x = \frac{1}{3}\left(1+2^{\frac{2-y}{3}}\right) \Rightarrow \text{Inversa}: y = \frac{1}{3}\left(1+2^{\frac{2-x}{3}}\right).$$

10) $f(x) = \begin{cases} e^{-x} & : x \leq 0 \\ mx+q & : x>0 \end{cases}$. La funzione è continua e derivabile $\forall x \neq 0$.

Per $x=0$: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} e^{-x} = 1 = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} mx+q = q \Rightarrow$

$\Rightarrow$ per essere continua in $x=0$ deve essere $q=1$.

Per la derivabilità : $f'(x) = \begin{cases} -e^{-x} & : x<0 \\ m & : x>0 \end{cases}$.

$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} (-e^{-x}) = -1 = \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} m = m \Rightarrow$

$\Rightarrow$ per essere derivabile in $x=0$ deve essere $m=-1$.

Quindi $f(x) = \begin{cases} e^{-x} & : x \leq 0 \\ -x+1 & : x>0 \end{cases}$.