Is Purity Necessary and Sufficient for Truncatability ?

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SUMMARY

In order to ensure the uniqueness of the Internal Rate of Return (I.R.R.) of a given project, two different proceedings can be found in the economic-financial literature: one based on the uniqueness of the I.R.R., later performed by the concept of purity, and another referring to the so-called Truncation Theorem.

The aim of this paper is to explain the connections between the purity of the I.R.R. of a given project and the truncatability of the project itself.

We will clarify a very important relation between purity and truncatability: if a given investment project has a "pure" I.R.R. i_0, then its Present Value is maximum compared with the Present Values of each of its shorter lives, when the rate of interest i belongs to an interval of the type $]-1; i_{max}[$, whith $i_0 \leq i_{max}$, and, as a main consequence, we will show that purity is necessary and sufficient for truncatability.

1 – PRELIMINARY DEFINITIONS

In this paper we will adopt the following definitions.

Definition 1 : We define as an investment project a vector of expected net outputs $\mathbb{A} = (a_0, a_1, ..., a_n)$ such that:

$$\begin{cases} a_0 < 0, \\ a_n \neq 0, \\ a_j \cdot a_k < 0, \text{ for at least one couple } j, k \end{cases} .$$

For every project $\mathbb{A}$ we consider its $n+1$ Present Value at time 0 functions:

$$\mathbb{V}_p(i) = \sum_{k=0}^{p} a_k \cdot (1+i)^{-k} =$$

$$= a_0 + \frac{a_1}{(1+i)} + \frac{a_2}{(1+i)^2} + ... + \frac{a_{p-1}}{(1+i)^{p-1}} + \frac{a_p}{(1+i)^p} \ ,$$

$$\forall\, p : 0 \leq p \leq n \ ,$$

and its $n+1$ Cumulated Value at time p functions:

$$\mathbb{S}_p(i) = \sum_{k=0}^{p} a_k \cdot (1+i)^{p-k} =$$

$= a_0 \cdot (1+i)^p + a_1 \cdot (1+i)^{p-1} + ... + a_{p-1} \cdot (1+i) + a_p \,,$
$\forall\, p : 0 \le p \le n \,.$

Obviously:

(1) $\mathbb{S}_p(i) = (1+i)^p \cdot \mathbb{V}_p(i) \,, \quad \forall\, p : 0 \le p \le n \,.$

Using these functions we have first of all:

Definition 2 : We define as an I.R.R. attached to project $\mathbb{A} = (a_0, a_1, ..., a_n)$ an interest rate i_0 such that:

$$\mathbb{S}_n(i_0) = \mathbb{V}_n(i_0) = 0 \;.$$

Following [14] and [15], we give then:

Definition 3 : We define a project $\mathbb{A} = (a_0, a_1, ..., a_n)$ as a “pure investment project" for a given rate of interest i_0 if the following inequalities are satisfied:

(2) $$\mathbb{S}_p(i_0) = \sum_{k=0}^{p} a_k \cdot (1+i_0)^{p-k} \;\le\; 0 \,, \qquad \forall\, p : 0 \le p \le n-1 \;.$$

or, by (1), if

$$\mathbb{V}_p(i_0) = \sum_{k=0}^{p} a_k \cdot (1+i_0)^{-k} \;\le\; 0 \,, \qquad \forall\, p : 0 \le p \le n-1 \;.$$

2 – PROPRIETIES OF PURE PROJECTS

As $\lim\limits_{i \to +\infty} \mathbb{S}_p(i) = -\infty \,,\ \forall\, p : 1 \le p \le n$, referring to [14], we can easily conclude:

Proposition 1 : Given an investment project $\mathbb{A} = (a_0, a_1, ..., a_n)$ then there exists an interest rate i_{min} such that:

$$\mathbb{S}_p(i) \le 0 \,, \qquad \forall\, i \in [i_{min}; +\infty[\,, \qquad \forall\, p : 0 \le p \le n-1 \,.$$

Clearly $i_{min} > -1$, and we can easily see that $i_{min} = -1$ if and only if a_n is the only positive cash-flow of the project.

In Appendix 1 we demonstrate the following:

Proposition 2 : Given an investment project $\mathbb{A} = (a_0, a_1, ..., a_n)$, $\mathbb{S}_p(i)$ is a non-positive decreasing function of i, $\forall\, i \in [i_{min}; +\infty[\,,\ \forall\, p : 1 \le p \le n-1$. Moreover, $\mathbb{S}(i)$ is a decreasing function in the same interval.

In addition, as demonstrated in reference [13] and [5], the following holds:

Proposition 3 : An I.R.R. i_0 attached to the project $\mathbb{A} = (a_0, a_1, ..., a_n)$ is unique if $\mathbb{A}$ is a pure investment project at that rate of interest i_0.

As a consequence, following reference [5], we also have:

Proposition 4 : If an investment project $\mathbb{A} = (a_0, a_1, ..., a_n)$ is pure in its I.R.R. i_0, then there exists a set:

$$\mathbb{B} : \left\{ \left(\Delta_{i-1}^{i} ; \Delta_{i}^{i} \right) : i = 1, ..., n \right\}$$

of n consecutive investment operations such that:

$$\Delta_0^1 = a_0 \ ;$$
$$\Delta_i^i = -(1+i_0) \cdot \Delta_{i-1}^i \ , \ \forall\, i : 1 \le i \le n \ ;$$
$$\Delta_i^i + \Delta_i^{i+1} = a_i \ , \ \forall\, i : 1 \le i \le n-1 \ ;$$
$$\Delta_n^n = a_n \ ;$$

which formally are (see [4]) :

$$\Delta_{i-1}^i = \mathbb{S}_{i-1}(i_0) \le 0 \ , \ \forall\, i : 1 \le i \le n \ ;$$
$$\Delta_i^i = -(1+i_0) \cdot \mathbb{S}_{i-1}(i_0) \ge 0 \ , \ \forall\, i : 1 \le i \le n \ .$$

So the I.R.R. i_0 is an interest rate uniformly applied to n consecutive investment operations into which a project, cumulated from time 0 to time n, can be uniquely decomposed (see figure below).

$$\begin{array}{llllllll}
\Delta_0^1 & \Rightarrow \Delta_1^1 & & & & & & \\
\Downarrow & \ \ + & & & & & & \\
\cdot & \Delta_1^2 & \Rightarrow \Delta_2^2 & & & & & \\
\cdot & \Downarrow & \ \ + & & & & & \\
\cdot & \cdot & \Delta_2^3 & \Rightarrow \Delta_3^3 & & & & \\
\cdot & \cdot & \Downarrow & \ \ + & & & & \\
\cdot & \cdot & \cdot & \Delta_3^4 & \Rightarrow ... & & & \\
\cdot & \cdot & \cdot & \Downarrow & ... & & & \\
\cdot & \cdot & \cdot & \cdot & ... & \Rightarrow \Delta_{n-2}^{n-2} & & \\
\cdot & \cdot & \cdot & \cdot & & \ \ + & & \\
\cdot & \cdot & \cdot & \cdot & & \Delta_{n-2}^{n-1} & \Rightarrow \Delta_{n-1}^{n-1} & \\
\cdot & \cdot & \cdot & \cdot & & \Downarrow & \ \ + & \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \Delta_{n-1}^{n} & \Rightarrow \Delta_n^n \\
\cdot & \cdot & \cdot & \cdot & & \cdot & \Downarrow & \ \ \Downarrow \\
 & & & & & & & \\
a_0 & a_1 & a_2 & a_3 & ... & a_{n-2} & a_{n-1} & a_n
\end{array}$$

It is demonstrable, by Proposition 2 (see also [9]), that the following hold:

Proposition 5 : An investment project $\mathbb{A} = (a_0, a_1, ..., a_n)$ satisfies conditions (2) if and only if $\mathbb{S}(i_{min}) \ge 0$.

Proposition 6 : A necessary condition for a project $\mathbb{A} = (a_0, a_1, ..., a_n)$ to have a pure I.R.R. i_0 is that $a_0 \cdot a_n < 0$.

3 − ANOTHER ECONOMIC INTERPRETATION OF PURITY

Let us now introduce the following functions of the rate of interest i:

$$\mathfrak{U}_p(i) = \sum_{k=0}^{p} a_{n-k} \cdot (1+i)^{k-p} =$$
$$= a_{n-p} + \frac{a_{n-p+1}}{(1+i)} + ... + \frac{a_{n-1}}{(1+i)^{p-1}} + \frac{a_n}{(1+i)^p} \ ,$$
$$\forall\, p : 0 \le p \le n-1 \,,$$

Present Value functions of the cash-flows $a_{n-p}, a_{n-p+1}, ..., a_n$ from time n to time $n-p$, and

$$\mathfrak{S}_p(i) = \sum_{k=0}^{p} a_{n-k} \cdot (1+i)^{k} =$$
$$= a_{n-p} \cdot (1+i)^p + a_{n-p+1} \cdot (1+i)^{p-1} + ... + a_{n-1} \cdot (1+i) + a_n \,,$$
$$\forall\, p : 0 \le p \le n-1 \,,$$

Cumulated Value functions of the cash-flows $a_{n-p}, a_{n-p+1}, ..., a_n$ from time $n-p$ to time n.

We clearly have $\mathbb{S}_n(i) = \mathfrak{S}_n(i)$ and $\mathbb{V}_n(i) = \mathfrak{U}_n(i)$.

After easy calculations we get:

$$\mathbb{V}(i) = \mathbb{V}_{p-1}(i) + (1+i)^{-p} \cdot \mathfrak{U}_{n-p}(i) \qquad \forall\, p : 1 \le p \le n \,,$$

and:

$$\mathbb{S}(i) = (1+i)^{n-p+1} \cdot \mathbb{S}_{p-1}(i) + \mathfrak{S}_{n-p}(i) \qquad \forall\, p : 1 \le p \le n \,,$$

from which, as $\mathfrak{S}_{n-p}(i) = (1+i)^{n-p} \cdot \mathfrak{U}_{n-p}(i)$, we get also:

$$\mathbb{S}(i) = (1+i)^{n-p} \cdot \mathfrak{U}_{n-p}(i) + (1+i)^{n-p+1} \cdot \mathbb{S}_{p-1}(i) \,,$$

and finally:

(3) $$(1+i)^{n-p} \cdot \mathfrak{U}_{n-p}(i) = \mathbb{S}(i) \ - \ (1+i)^{n-p+1} \cdot \mathbb{S}_{p-1}(i) \,.$$

Let us now consider the equation $\mathbb{V}(i) = 0$, in which we put $y = \dfrac{1}{(1+i)}$, to have:

$$a_n \cdot y^n + a_{n-1} \cdot y^{n-1} + ... + a_1 \cdot y + a_0 = 0 \ .$$

If the equation $\mathbb{V}(i) = 0$ has a solution i_0, we put $(1+i_0)^{-1} = y_0$ to obtain:

$$a_n \cdot y^n + a_{n-1} \cdot y^{n-1} + ... + a_1 \cdot y + a_0 = \mathbb{Q}(y) \cdot (y - y_0) \,.$$

Under the hypotesis $a_n > 0$, if the coefficients of the polynomial $\mathbb{Q}(y)$ are all non-negative like the first, a_n, then, for Descartes' rule of signs applied to polynomial $\mathbb{Q}(y)$, the root i_0 is unique.

But it is easy to see that the coefficients of the polynomial $\mathbb{Q}(y)$ are given by the $\mathfrak{U}_p(i_0)$, $\forall\, p : 0 \le p \le n-1$, so the following holds:

Proposition 7 : Given a project $\mathbb{A} = (a_0, a_1, ..., a_n)$ with an I.R.R. i_0, if the following inequalities are satisfied:

$$(4) \qquad \mathfrak{U}_p(i_0) = \sum_{k=0}^{p} a_{n-k} \cdot (1+i_0)^{k-p} \ \ge\ 0\,, \qquad \forall\, p : 0 \le p \le n-1\,,$$

then the I.R.R. i_0 is unique.

As $\mathfrak{S}_p(i) = (1+i)^p \cdot \mathfrak{U}_p(i)$, (4) is equivalent to

$$\mathfrak{S}_p(i_0) = \sum_{k=0}^{p} a_{n-k} \cdot (1+i_0)^{k} \ \ge\ 0\,, \qquad \forall\, p : 0 \le p \le n-1\,.$$

Using (3), as $\mathbb{S}(i_0) = 0$, we have immediately:

Proposition 8 : Conditions (2) are satisfied if and only if conditions (4) are satisfied.

We can get another economic interpretation of purity, because we can demonstrate, as in Proposition 4, that the following holds:

Proposition 9 : Given a project $\mathbb{A} = (a_0, a_1, ..., a_n)$, which is pure in its I.R.R. i_0, there exists a set

$$\mathbb{D} : \left\{ \left(\Theta^i_{i-1};\Theta^i_i\right) : i = 1, ..., n \right\}$$

of n consecutive financing operations such that:

$$\Theta^1_0 = a_n \ ;$$
$$\Theta^i_i = \, -(1+i_0)^{-1} \cdot \Theta^i_{i-1} \ , \ \forall\, i : 1 \le i \le n\,;$$
$$\Theta^i_i + \Theta^{i+1}_i = a_{n-i} \ , \ \forall\, i : 1 \le i \le n-1\,;$$
$$\Theta^n_n = a_0\,;$$

which formally are:

$$\Theta^i_{i-1} = \mathfrak{U}_{i-1}(i_0) \ge 0 \ , \ \forall\, i : 1 \le i \le n\,;$$
$$\Theta^i_i = \, -\frac{1}{(1+i_0)} \cdot \mathfrak{U}_{i-1}(i_0) \le 0 \ , \ \forall\, i : 1 \le i \le n\,.$$

So, the I.R.R. i_0 is an interest rate uniformly applied to n consecutive financing operations into which a project, discounted from time n to time $n-p$, $p : 0 \le p \le n$, can be uniquely decomposed.

It is easy to see, by the definitions and as $\mathbb{S}(i_0) = 0$, that the following relation holds:

$$\Theta^{n-i+1}_{n-i} = \, -(1+i_0) \cdot \Delta^i_{i-1}\,.$$

If $a_n > 0$, we have $\lim\limits_{i \to -1^+} \mathfrak{U}_p(i) = \, +\infty$, $\forall\, p : 1 \le p \le n$, so it is easily deducible that:

Proposition 10 : If in the project $\mathbb{A} = (a_0, a_1, ..., a_n)$ is $a_n > 0$, then an interest rate i_{max} exists such that:

$$\mathfrak{U}_p(i) \geq 0\,, \quad \forall\, i \in \,]-1; i_{max}[\,, \quad \forall\, p:\; 0 \leq p \leq n-1\,.$$

Proposition 11 : If in the project $\mathbb{A} = (a_0, a_1, ..., a_n)$ is $a_n > 0$, then $\mathfrak{U}_p(i)$ is a non-negative decreasing function of i, $\forall\, i \in \,]-1;_{max}[$, $\forall\, p: 1 \leq p \leq n-1$. Moreover, $\mathbb{V}(i)$ is a decreasing function in the same interval.

The Proof is given in Appendix 2.

It is also possible to demonstrate (see [9]) that:

Proposition 12 : A project $\mathbb{A} = (a_0, a_1, ..., a_n)$ satisfies conditions (4) if and only if $\mathfrak{U}_n(i_{max}) = \mathbb{V}(i_{max}) \leq 0$.

Finally, see [9], the following holds:

Proposition 13 : A project $\mathbb{A} = (a_0, a_1, ..., a_n)$ has a pure I.R.R. i_0 if and only if $i_{min} \leq i_{max}$. In this case we also have that : $i_{min} \leq i_0 \leq i_{max}$.

On the contrary, when a project has a unique I.R.R. which is not pure or it has more than one I.R.R., $i_1, i_2, ..., i_k$, in this case it is easily deducible that $i_{min} > i_{max}$ and that $i_{max} \leq i_1, i_2, ..., i_k \leq i_{min}$.

4 – THE TRUNCATION THEOREM

Starting from the behavioural hypothesis that the investor might choose the duration of a given project, neglecting scrap values, two different procedures for choosing the optimal duration can be found in the economic literature.

The first, due to P. H. Karmel (see [6] and [5]), suggests to choose that particular duration to which the maximum I.R.R. is attached, and in article [6] it has been proved that if an investor chooses as the optimum duration of a project that particular duration to which the maximum I.R.R. is attached, then the I.R.R. connected to this duration is unique.

S. Gronchi demonstrates in [5] that, doing so, such an I.R.R. is pure too.

Many Authors instead, see [1], [3], [16] and [17], object that the best truncation criterion is to pursue the maximisation of the Present Value.

In fact, K. J. Arrow and D. Levhari in [1] state that: "...choosing a truncation period so as to maximise the I.R.R. is not using the proper criterion for the selection of a truncation period ... one should choose that investment which maximises Present Value, using the given rates of discount ... the aim in the choice of a truncation period should be that of maximising the Present Value of the investment project."

J. F. Wright has demonstrated in [16] that if, at a given rate of discount, from the possible lenghts of life of the project the enterpreneur selects that with the greatest Present Value, then the Present Value function is decreasing at that rate.

Incidentally he also demonstrates that if at a rate i the Present Value of the project is maximum compared to every shorter life of its, then at this rate i the project is also pure.

This result has been generalizes in two ways, the first in [3], and furthermore in [11] and [12], where it is supposed that the rate can vary during the periods, and the second in [1], where K. J. Arrow and D. Levhari say to have proved that if, with a given constant rate of discount, we choose the truncation period so as to maximise the Present Value of the project, then the I.R.R. of the truncated project is unique.

However it has been proved in [10] that Karmel's procedure and Arrow and Levhari's one lead to the same numerical result.

To be more precise, Arrow and Levhari prove that if the life of the project is optimally chosen, then the maximised Present Value of the project is a monotonic decreasing function of the rate of interest.

But they also say that "... $\psi(i) = \max_p \mathbb{V}_p(i)$, the Present Value of the best truncated project, is a decreasing monotonic function of i."

This is not exact, as $\psi(i)$ is not, generally, the Present Value of the best truncated project; it is the maximum, varying i, among the $\mathbb{V}_p(i)$, and we shall demonstrate that this coincidence is true if and only if the project has a pure I.R.R. .

This incorrectness is also found in [17], where J. F. Wright says:
"if truncatability (costless extricability) holds, then the Present Value of a project for its whole life will be greater than its Present Value for some shorter life..." and "... thus the simple time-profile ensured by truncatability is identical with that implied by purity. But a project might happen to have such a profile even though it cannot be truncated. Thus truncatability is a sufficient rather than a necessary condition; and consequently less fundamental than purity."

5 – TRUNCATABILITY AND PURITY

The following Proposition could clarify the above-mentioned ambiguities.

In fact, we can prove that:

Proposition 14 : If the project $\mathbb{A} = (a_0, a_1, ..., a_n)$ has a pure I.R.R. i_0, then $\mathbb{V}(i)$ is maximum compared to all $\mathbb{V}_p(i)$, $0 \leq p \leq n-1$, $\forall\, i \in]-1; i_{max}[$.

Proof : Let us say that $\mathbb{V}(i) - \mathbb{V}_{n-p-1}(i) \geq 0$, $\forall\, p : 0 \leq p \leq n-1$.

In fact we have:

$$\mathbb{V}(i) - \mathbb{V}_{n-p-1}(i) = \frac{a_{n-p}}{(1+i)^{n-p}} + ... + \frac{a_{n-1}}{(1+i)^{n-1}} + \frac{a_n}{(1+i)^n} =$$
$$= \frac{1}{(1+i)^{n-p}} \cdot \left[a_{n-p} + ... + \frac{a_{n-1}}{(1+i)^{p-1}} + \frac{a_n}{(1+i)^p} \right] =$$
$$= \frac{1}{(1+i)^{n-p}} \cdot \mathfrak{U}_p(i) \,.$$

But the I.R.R. of the project is pure, and so, by Proposition 11 we see that, $\forall\, p : 0 \leq p \leq n-1$, $\mathfrak{U}_p(i)$ is a non-negative function, $\forall\, i$: $i \in \,]-1; i_{max}[$ and so we have the Proof.

If a project is pure in its I.R.R., then its Present Value is maximum compared with the Present Values of all its truncations, when the rate of interest varies in an interval into which its I.R.R. lies; so the Present Value $\mathbb{V}(i)$ is the same as the maximised Present Value and Arrow and Levhari's assertions are true.

The best truncation, if the project is pure in its I.R.R., is the whole project so we can conclude that pureness is necessary and sufficient for truncatability and that, using Karmel's procedure, from every project we can draw out a sub-project, which can be the project itself, the I.R.R. of which is pure.

To conclude, we also have the

Proposition 15 : If the project $\mathbb{A} = (a_0, a_1, ..., a_n)$ has a pure I.R.R. i_0, then $\mathfrak{S}(i)$ is minimum compared with all $\mathfrak{S}_p(i)$, $0 \leq p \leq n-1$, $\forall\, i \in [i_{min}; +\infty[$.

Proof : Let us say that $\mathfrak{S}(i) - \mathfrak{S}_{n-p-1}(i) \leq 0$, $\forall\, p : 0 \leq p \leq n-1$.

In fact we have:

$$\mathfrak{S}(i) - \mathfrak{S}_{p-1}(i) =$$
$$= a_{n-p} \cdot (1+i)^p + ... + a_1 \cdot (1+i)^{n-1} + a_0 \cdot (1+i)^n =$$
$$= (1+i)^p \cdot [\, a_{n-p} + a_{n-p-1}(1+i) + ... + a_0 \cdot (1+i)^{n-p} \,] =$$
$$= (1+i)^p \cdot \mathbb{S}_{n-p}(i) \,.$$

But, by the hypoteses and as (2) holds, we have the Proof.

However it is possible for a project to have a Present Value maximum for all acceptable rates $i : -1 < i < +\infty$.

It is easy to prove that it is possible if and only if a_0 is the only negative cash-flow. In this case we put $i_{max} = +\infty$.

So for the uniperiodal project $\mathbb{A} = (a_0, a_1)$, $(a_0 < 0,\, a_1 > 0)$ we have $i_{min} = -1$ and $i_{max} = +\infty$, and this propriety extends only to the projects of the type $\mathbb{A} = (a_0, 0, ..., 0, a_n)$.

APPENDIX 1

Proof of Proposition 2 : If $i \in [i_{min};\ +\infty[$, we have:

$\mathbb{S}_0(i) = a_0 < 0\,;$

$\mathbb{S}_1(i) = a_0 \cdot (1+i) + a_1\,,$

for which $\mathbb{S}_1(i_{min}) \leq 0$ and $\mathbb{S}_1'(i) = a_0 < 0$, and so $\mathbb{S}_1(i)$ is a non-positive decreasing function $\forall\, i \in [i_{min};\ +\infty[$.

By induction, suppose $\mathbb{S}_{p-1}(i)$ is a non-positive decreasing function $\forall\, i \in [i_{min};\ +\infty[$.

Then:

$\mathbb{S}_p(i) = \mathbb{S}_{p-1}(i) \cdot (1+i) + a_p$, with

$\mathbb{S}_p(i_{min}) \leq 0$, and

$\mathbb{S}_p'(i) = \mathbb{S}_{p-1}(i) + (1+i) \cdot \mathbb{S}_{p-1}'(i)\,,$

so $\mathbb{S}_p(i)$ is decreasing because $\mathbb{S}_{p-1}(i)$ is non-positive and decreasing.

By induction, it follows Proposition 2, $\forall\, p : 1 \leq p \leq n-1$.

But $\mathbb{S}(i) = \mathbb{S}_{n-1}(i) \cdot (1+i) + a_n$, so

$\mathbb{S}'(i) = \mathbb{S}_{n-1}(i) + (1+i) \cdot \mathbb{S}_{n-1}'(i) \leq 0\,,\ \ \forall\, i \in [i_{min}; +\infty[$.

APPENDIX 2

Proof of Proposition 11 : If $i \in]-1; i_{max}[$, we have:

$\mathfrak{U}_0(i) = a_n > 0$;

$\mathfrak{U}_1(i) = a_n \cdot (1+i)^{-1} + a_{n-1}$,

for which $\mathfrak{U}_1(i_{max}) \geq 0$ and $\mathfrak{U}_1'(i) = -a_n \cdot (1+i)^{-2} < 0$, and so $\mathfrak{U}_1(i)$ is a non-negative decreasing function $\forall\, i \in]-1; i_{max}[$.

By induction, suppose $\mathfrak{U}_{p-1}(i)$ is a non-negative decreasing function $\forall\, i \in]-1; i_{max}[$.

Then :

$\mathfrak{U}_p(i) = \mathfrak{U}_{p-1}(i) \cdot (1+i)^{-1} + a_{n-p}$, with

$\mathfrak{U}_p(i_{max}) \geq 0$, and

$\mathfrak{U}_p'(i) = -(1+i)^{-2} \cdot \mathfrak{U}_{p-1}(i) + (1+i)^{-1} \cdot \mathfrak{U}_{p-1}'(i)$,

so $\mathfrak{U}_p(i)$ is decreasing because $\mathfrak{U}_{p-1}(i)$ is non-negative and decreasing .

By induction, Proposition 11 holds $\forall\, p : 1 \leq p \leq n-1$.

But $\mathbb{V}(i) = \mathfrak{U}(i) = \mathfrak{U}_{n-1}(i) \cdot (1+i)^{-1} + a_0$, so

$\mathbb{V}'(i) = -(1+i)^{-2} \cdot \mathfrak{U}_{n-1}(i) + (1+i)^{-1} \cdot \mathfrak{U}_{n-1}'(i) \leq 0$, $\forall\, i \in]-1; i_{max}[$.

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