Compito di Analisi Matematica del 9/1/2017 $\boxed{\text{CAM1}}$

IM1) $(x-i)^3 = \frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{1-1-2i}{1+1} = -i \Rightarrow x-i = \sqrt[3]{-i}$.

$-i = 1\left(\cos\frac{3}{2}\pi + i\,\text{sen}\frac{3}{2}\pi\right) \Rightarrow \sqrt[3]{-1} = 1\cdot\left(\cos\left(\frac{\pi}{2}+k\cdot\frac{2\pi}{3}\right) + i\,\text{sen}\left(\frac{\pi}{2}+k\cdot\frac{2\pi}{3}\right)\right);\ 0 \leq k \leq 2.$

Per $k=0$: $x-i = \cos\frac{\pi}{2} + i\,\text{sen}\frac{\pi}{2} \Rightarrow x-i = i \Rightarrow x = 2i$;

Per $k=1$: $x-i = \cos\frac{7}{6}\pi + i\,\text{sen}\frac{7}{6}\pi \Rightarrow x-i = -\frac{\sqrt{3}}{2} - \frac{1}{2}i \Rightarrow x = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$;

Per $k=2$: $x-i = \cos\frac{11}{6}\pi + i\,\text{sen}\frac{11}{6}\pi \Rightarrow x-i = \frac{\sqrt{3}}{2} - \frac{1}{2}i \Rightarrow x = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

IM2) $f(x;y) = |x|^k\cdot\sqrt[10]{|y|^3}$ .. $\lim_{(x;y)\to(0;0)} f(x;y) = 0 = f(0;0)$: $f \in C(0;0)$.

$\frac{\partial f}{\partial x}(0;0) = \lim_{h\to 0}\frac{|0+h|^k\cdot\sqrt[10]{0} - 0}{h} = 0$; $\frac{\partial f}{\partial y}(0;0) = \lim_{h\to 0}\frac{0^k\cdot\sqrt[10]{|h|^3} - 0}{h} = 0$.

Per la differenziabilità deve risultare $\lim_{(x;y)\to(0;0)}\frac{f(x;y) - f(0;0) - \nabla f(0;0)(x-0;y-0)}{\sqrt{x^2+y^2}} = 0 \Rightarrow$

$\Rightarrow \lim_{(x;y)\to(0;0)}\frac{|x|^k\cdot\sqrt[10]{|y|^3} - 0 - 0}{\sqrt{x^2+y^2}} \Rightarrow \lim_{\rho\to 0}\frac{\rho^k\cdot|\cos^k\vartheta|\cdot\sqrt[10]{|\rho\cdot\text{sen}\,\vartheta|^3}}{\rho} =$

$= \lim_{\rho\to 0}\rho^{k-1+\frac{3}{10}}\cdot|\cos^k\vartheta|\cdot\sqrt[10]{|\text{sen}\,\vartheta|^3} = 0$ se $k-1+\frac{3}{10} > 0 \Rightarrow k > 1-\frac{3}{10} = \frac{7}{10}$.

Se $k > \frac{7}{10}$ la convergenza è uniforme $\left(\left||\cos^k\vartheta|\cdot\sqrt[10]{|\text{sen}\,\vartheta|^3}\right| < 1\right)$ e la funzione risulta differenziabile.

IM3) $\begin{cases} f(x;y;z) = xyz + x^2y - xz^2 - yz = 0 \\ g(x;y;z) = e^{x-y} - 2e^{y-z} + e^{z-x} = 0 \end{cases}$; $P \equiv (1;1;1)$; $\begin{cases} f(P) = 1+1-1-1 = 0 \\ g(P) = 1-2+1 = 0 \end{cases}$

$\frac{\partial(f;g)}{\partial(x;y;z)} = \begin{Vmatrix} yz+2xy-z^2 & xz+x^2-z & xy-2xz-y \\ e^{x-y}-e^{z-x} & -e^{x-y}-2e^{y-z} & 2e^{y-z}+e^{z-x} \end{Vmatrix} \Rightarrow \frac{\partial(f;g)}{\partial(x;y;z)}(1;1;1) = \begin{Vmatrix} 2 & 1 & -2 \\ 0 & -3 & 3 \end{Vmatrix}$

Dato che $\begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix} = 3-6 = -3 \neq 0$ esiste funzione implicita $x \to (y(x); z(x))$.

$\boxed{\text{CAM2}}$

$$\frac{dy}{dx} = -\frac{\begin{vmatrix}2&-2\\0&3\end{vmatrix}}{\begin{vmatrix}1&-2\\-3&3\end{vmatrix}} = -\frac{6}{-3} = 2;\quad \frac{dz}{dx} = -\frac{\begin{vmatrix}1&2\\-3&0\end{vmatrix}}{\begin{vmatrix}1&-2\\-3&3\end{vmatrix}} = -\frac{6}{-3} = 2.\quad X_0' = (2;2).$$

Equazione retta tangente: $x \to X_0 + x\cdot X_0' \Rightarrow x \to (1;1) + x\cdot(2;2) = (1+2x;1+2x)$.

IM4) $f(x;y) = xye^{x-y}$ è funzione differenziabile due volte quindi:

$\mathcal{D}^2_{v,w} f(1;1) = v\cdot H(1;1)\cdot w^T$. $\nabla f(x;y) = \left((y+xy)e^{x-y};\ (x-xy)e^{x-y}\right)$.

$$H(x;y) = \left\| \begin{matrix} ye^{x-y} + (y+xy)e^{x-y} & (1+x)e^{x-y} - (y+xy)e^{x-y} \\ (1-y)e^{x-y} + (x-xy)e^{x-y} & -xe^{x-y} - (x-xy)e^{x-y} \end{matrix} \right\| = \left\| \begin{matrix} (2y+xy)e^{x-y} & (1+x-y-xy)e^{x-y} \\ (1+x-y-xy)e^{x-y} & (xy-2x)e^{x-y} \end{matrix} \right\|$$

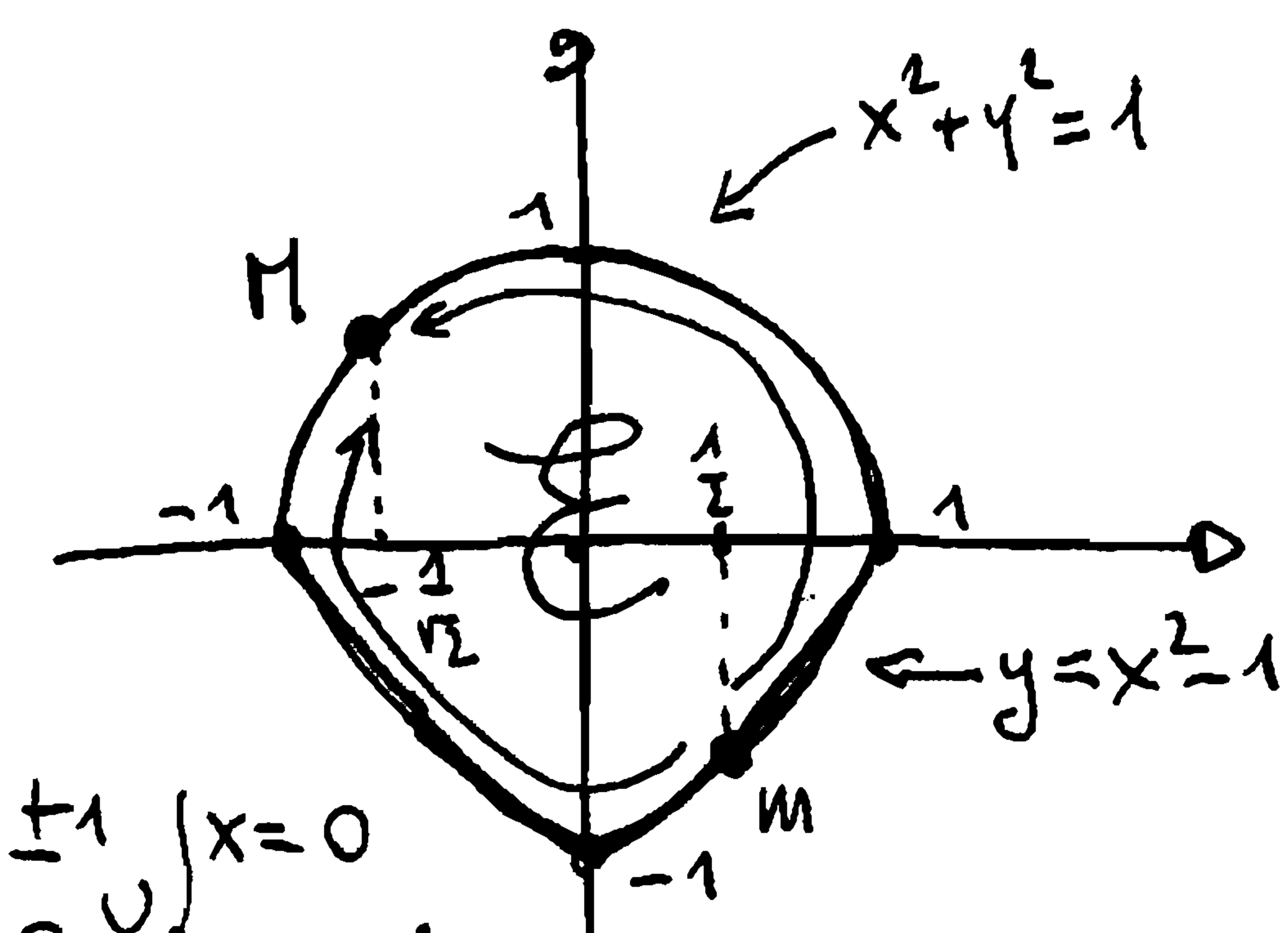
$$H(1;1) = \left\| \begin{matrix} 3 & 0 \\ 0 & -1 \end{matrix} \right\|.\quad \mathcal{D}^2_{v,w} f(1;1) = \left\| \begin{matrix} \cos\alpha & \operatorname{sen}\alpha \end{matrix} \right\| \cdot \left\| \begin{matrix} 3 & 0 \\ 0 & -1 \end{matrix} \right\| \cdot \left\| \begin{matrix} \operatorname{sen}\alpha \\ \cos\alpha \end{matrix} \right\| =$$

$$= \left\| \begin{matrix} \cos\alpha & \operatorname{sen}\alpha \end{matrix} \right\| \cdot \left\| \begin{matrix} 3\operatorname{sen}\alpha \\ -\cos\alpha \end{matrix} \right\| = 3\operatorname{sen}\alpha\cos\alpha - \operatorname{sen}\alpha\cos\alpha = 2\operatorname{sen}\alpha\cos\alpha = \operatorname{sen}2\alpha.$$

Quindi $\mathcal{D}^2_{v,w} f(1;1)$ sarà massima se $\operatorname{sen}2\alpha = 1 \Rightarrow 2\alpha = \frac{\pi}{2}$ o $2\alpha = \frac{5}{2}\pi \Rightarrow$

$\Rightarrow \alpha = \frac{\pi}{4}$ oppure $\alpha = \frac{5}{4}\pi$.

IIM1) $\begin{cases} \text{Max/min } f(x;y) = y - x \\ \text{s.v.} \begin{cases} x^2+y^2-1 \leq 0 \\ x^2 - y - 1 \leq 0 \end{cases} \end{cases}$. $f(x;y)$ continua, $\mathcal{E}$ limitato e chiuso, vincoli qualificati.



$$\begin{cases} x^2+y^2=1 \\ y = x^2-1 \end{cases} \Rightarrow \begin{cases} x^2+y^2=1 \\ x^2 = y+1 \end{cases} \Rightarrow \begin{cases} y+1+y^2=1 \\ y=x^2-1 \end{cases} \Rightarrow \begin{cases} y(y+1)=0 \\ y=x^2-1 \end{cases} \Rightarrow \begin{cases} x=\pm1 \\ y=0 \end{cases} \cup \begin{cases} x=0 \\ y=-1 \end{cases}.$$

$$\Lambda = y - x - \lambda_1(x^2+y^2-1) - \lambda_2(x^2-y-1).$$

<u>Caso $\lambda_1 = \lambda_2 = 0$</u>

$\begin{cases} \Lambda'_x = -1 \neq 0 \\ \Lambda'_y = 1 \neq 0 \end{cases}$ non ci sono soluzioni.

CAM3

Caso $\lambda_1 \neq 0; \lambda_2 = 0$

$$\begin{cases} \Lambda'_x = -1-2\lambda_1 x = 0 \\ \Lambda'_y = 1-2\lambda_1 y = 0 \\ x^2+y^2=1 \\ y \geq x^2-1 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2\lambda_1} \\ y = \frac{1}{2\lambda_1} \\ \frac{1}{4\lambda_1^2}+\frac{1}{4\lambda_1^2}=1 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2\lambda_1} \\ y = \frac{1}{2\lambda_1} \\ \lambda_1^2 = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{\sqrt{2}} \\ y = \frac{1}{\sqrt{2}} \\ \lambda_1 = \frac{1}{\sqrt{2}} \end{cases} \cup \begin{cases} x = \frac{1}{\sqrt{2}} \\ y = -\frac{1}{\sqrt{2}} \\ \lambda_1 = -\frac{1}{\sqrt{2}} \end{cases}$$

$y \geq x^2-1$: in $\left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right)$: $-\frac{1}{\sqrt{2}} \geq \frac{1}{2}-1 = -\frac{1}{2}$: FALSA. in $\left(-\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$: $\frac{1}{\sqrt{2}} \geq \frac{1}{2}-1 = -\frac{1}{2}$: VERA

Quindi il punto $\left(-\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$ potrebbe essere un punto di Max mentre $\left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}}\right) \notin E$.

Caso $\lambda_1 = 0; \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = -1-2\lambda_2 x = 0 \\ \Lambda'_y = 1+\lambda_2 = 0 \\ y = x^2-1 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2\lambda_2} = \frac{1}{2} \\ \lambda_2 = -1 \\ y = \frac{1}{4}-1 = -\frac{3}{4} \end{cases}. \quad x^2+y^2 \leq 1 : \frac{1}{4}+\frac{9}{16} = \frac{13}{16} \leq 1 \text{ VERA}.$$

Da $\lambda_2 = -1 \Rightarrow \left(\frac{1}{2}; -\frac{3}{4}\right)$ potrebbe essere un punto di MIN.

Caso $\lambda_1 \neq 0; \lambda_2 \neq 0$

$$\begin{cases} \Lambda'_x = -1-2\lambda_1 x - 2\lambda_2 x = 0 \\ \Lambda'_y = 1-2\lambda_1 y + \lambda_2 = 0 \\ \begin{cases} x^2+y^2=1 \\ y = x^2-1 \end{cases} \Rightarrow \begin{cases} x=-1 \\ y=0 \end{cases} \cup \begin{cases} x=0 \\ y=-1 \end{cases} \cup \begin{cases} x=1 \\ y=0 \end{cases} \end{cases} \Rightarrow \begin{cases} x=-1 \\ y=0 \\ -1+2\lambda_1+2\lambda_2=0 \\ 1-0+\lambda_2=0 \end{cases} \Rightarrow \begin{cases} x=-1 \\ y=0 \\ \lambda_1 = \frac{1}{2}(1-2\lambda_2) = \frac{3}{2} > 0 \\ \lambda_2 = -1 < 0 : \text{NULLA} \end{cases}.$$

$$\begin{cases} x=0 \\ y=-1 \\ -1+0+0=0 : \text{impossibile} \\ 1+2\lambda_1+\lambda_2=0 \end{cases}; \quad \begin{cases} x=1 \\ y=0 \\ -1-2\lambda_1-2\lambda_2=0 \\ 1+\lambda_2=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=0 \\ \lambda_1 = \frac{1}{2}\cdot(-1-2\lambda_2) = \frac{1}{2} \\ \lambda_2 = -1 \quad : \text{NULLA} \end{cases}$$

Per il Teorema di Weierstrass $\left(-\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}\right)$ è il punto di Massimo assoluto mentre $\left(\frac{1}{2}; -\frac{3}{4}\right)$ è il punto di minimo assoluto.

L'eventuale (non necessaria) analisi sul bordo conferma questa conclusione.

II M2) $\begin{cases} y'-xy = x \\ y(0)=1 \end{cases} \Rightarrow y' = xy+x = x(y+1) \Rightarrow \frac{1}{y+1}\cdot y' = x$ (posto $y+1 \neq 0 \Rightarrow y \neq -1$)

La funzione $y=-1$ è una soluzione del problema $y' = x(y+1)$.

CAM4

$\int \frac{1}{y+1}\,dy = \int x\,dx + K \Rightarrow \log|y+1| = \frac{x^2}{2} + K \Rightarrow |y+1| = e^{\frac{x^2}{2}+K} \Rightarrow$

$\Rightarrow |y+1| = m \cdot e^{\frac{x^2}{2}}$ $\quad (m = e^K) \Rightarrow y = m \cdot e^{\frac{x^2}{2}} - 1 \quad (m \in \mathbb{R})$.

$y(0) = 1 \Rightarrow 1 = m \cdot e^0 - 1 \Rightarrow m = 2$. Soluzione P. Cauchy: $y = 2e^{\frac{x^2}{2}} - 1$.

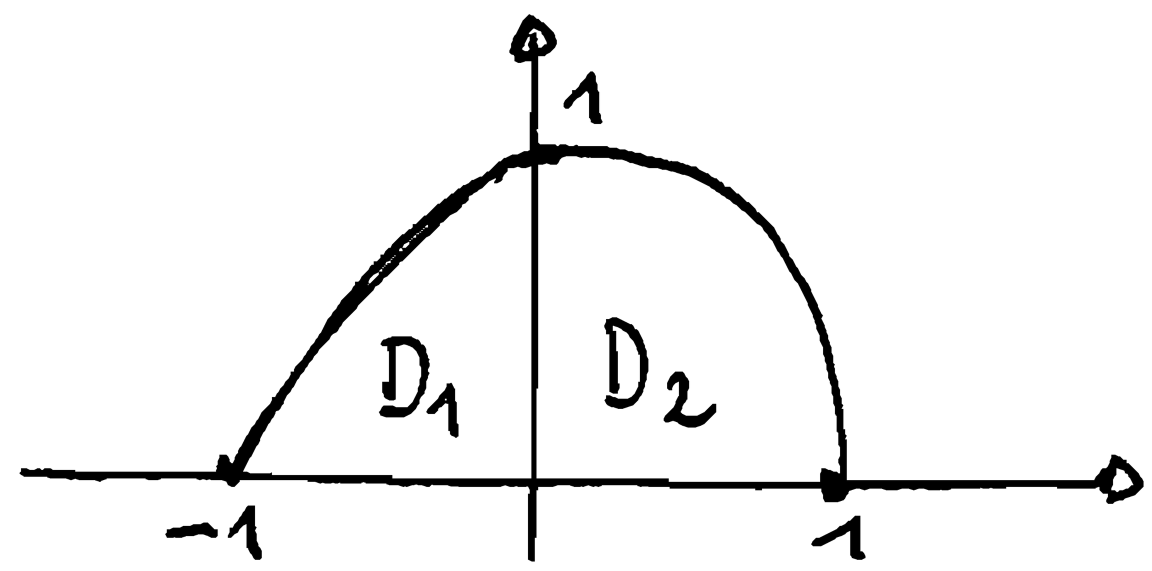
II M3) $\begin{cases} x' = y - 1 \\ y' = x + t \end{cases} \Rightarrow \begin{cases} y = x' + 1 \\ x'' + 0 = x + t \end{cases} \Rightarrow x'' - x = t \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$.

Soluzione $x(t)$ dell'omogenea: $x(t) = c_1 e^t + c_2 e^{-t}$. Preso $x_0(t) = at + b \Rightarrow$

$\Rightarrow x_0'(t) = a$; $x_0''(t) = 0 \Rightarrow 0 - at - b = t \Rightarrow a = -1$ e $b = 0$. Quindi

$x(t) = c_1 e^t + c_2 e^{-t} - t$. Ma $y = x' + 1 \Rightarrow y(t) = c_1 e^t - c_2 e^{-t} - 1 + 1 = c_1 e^t - c_2 e^{-t}$.

II M4) $\iint_D xy\,dx\,dy = \iint_{D_1} xy\,dx\,dy + \iint_{D_2} xy\,dx\,dy$



$\iint_{D_1} xy\,dx\,dy = \int_{-1}^{0} \int_0^{1-x^2} xy\,dy\,dx = \int_{-1}^{0} \left(x \cdot \frac{y^2}{2} \right|_0^{1-x^2} dx =$

$= \int_{-1}^{0} \frac{x(1-x^2)^2}{2} - 0\,dx = -\frac{1}{4} \cdot \int_{-1}^{0} (-2x)(1-x^2)^2\,dx = -\frac{1}{4} \cdot \left(\frac{1}{3}(1-x^2)^3 \right|_{-1}^{0} = -\frac{1}{4} \cdot \frac{1}{3} \cdot (1-0) = -\frac{1}{12}$.

$\iint_{D_2} xy\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^1 \rho^2 \cos\vartheta \operatorname{sen}\vartheta \cdot \rho\,d\rho\,d\vartheta = \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \cos\vartheta \operatorname{sen}\vartheta\,d\rho\,d\vartheta =$

$= \int_0^{\frac{\pi}{2}} \left(\frac{\rho^4}{4} \right|_0^1 \cos\vartheta \operatorname{sen}\vartheta\,d\vartheta = \frac{1}{4} \int_0^{\frac{\pi}{2}} \cos\vartheta \operatorname{sen}\vartheta\,d\vartheta = \frac{1}{4} \left(\frac{1}{2} \operatorname{sen}^2\vartheta \right|_0^{\frac{\pi}{2}} = \frac{1}{8}(1-0) = \frac{1}{8}$.

Quindi $\iint_D f(x;y)\,dx\,dy = \frac{1}{8} - \frac{1}{12} = \frac{3-2}{24} = \frac{1}{24}$.