TASK MATHEMATICS for ECONOMIC APPLICATIONS
09/01/2017

I M 1) From $(x-i)^3 = i$ we get $x - i = \sqrt[3]{i}$ and so $x = i + \sqrt[3]{i}$. To find the three cubic roots of i remember that $i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$, and so:

$\sqrt[3]{i} = \cos\left(\frac{\pi}{6} + \frac{2}{3}k\pi\right) + i\sin\left(\frac{\pi}{6} + \frac{2}{3}k\pi\right)$ with $k = 0, 1, 2$. The three solutions are:

$$x_0 = i + \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right) = i + \frac{\sqrt{3}}{2} + \frac{1}{2}i = \frac{\sqrt{3}}{2} + \frac{3}{2}i\,;$$

$$x_1 = i + \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = i - \frac{\sqrt{3}}{2} + \frac{1}{2}i = -\frac{\sqrt{3}}{2} + \frac{3}{2}i\,;$$

$$x_2 = i + \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = i + 0 - i = 0\,.$$

I M 2) The characteristic polinomial of $\mathbb{A}$ is $p_{\mathbb{A}}(\lambda) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & -1 & 1-\lambda \end{vmatrix} =$

$$= (2-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (2-\lambda)\left((1-\lambda)^2 - 1\right) = (2-\lambda)\left(\lambda^2 - 2\lambda\right);$$

from $p_{\mathbb{A}}(\lambda) = 0$ we have the three eigenvalues: $\lambda_1 = 0, \lambda_2 = \lambda_3 = 2$.

To find eigenvectors corresponding to the eigenvalue $\lambda_1 = 0$ we must solve the linear homogeneous system $|\mathbb{A} - 0\,\mathbb{I}| \cdot \mathbb{X} = \mathbb{O}$ to get:

$$\left\|\begin{matrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\| \Rightarrow \begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_2 = 0 \\ x_1 - x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -x_1 \\ x_2 = 0 \end{cases}$$ and so the eigenvectors corresponding to $\lambda_1 = 0$ are $\mathbb{X}_0 = (x; 0; -x) = x\,(1; 0; -1)$.

To find eigenvectors corresponding to the eigenvalue $\lambda_{2,3} = 2$ we must solve the linear homogeneous system $|\mathbb{A} - 2\,\mathbb{I}| \cdot \mathbb{X} = \mathbb{O}$ to get:

$$\left\|\begin{matrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{matrix}\right\| \cdot \left\|\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\| \Rightarrow \begin{cases} -x_1 - x_2 + x_3 = 0 \\ 0 = 0 \\ x_1 - x_2 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = x_1 \\ x_2 = 0 \end{cases}$$ and so the eigenvectors corresponding to $\lambda_{2,3} = 2$ are $\mathbb{X}_2 = (x; 0; x) = x\,(1; 0; 1)$.

To check if $\mathbb{A}$ is or not a diagonalizable matrix we must check if $m^g_\lambda = m^a_\lambda$ (the geometric multiplicity of $\lambda = 2$, the dimention of $\mathcal{ES}_{\lambda=2}$, the eigenspace associated to eigenvalue 2, is equal or not to 2, its algebraic multiplicity).

Since $m^g_2 = 3 - \text{Rank}(\mathbb{A} - 2\,\mathbb{I})$ and $\text{Rank}(\mathbb{A} - 2\,\mathbb{I}) = \text{Rank}\left\|\begin{matrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{matrix}\right\| = 2$ the matrix is not diagonalizable.

I M 3) The dimension of the Image of a linear map is equal to the Rank of the matrix $\mathbb{A}$ associated to the map; using elementary operations on the lines of the matrix we get:

$$\left\|\begin{matrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & -2 & k \end{matrix}\right\| \underset{R_3 - 2R_1}{\overset{R_2 - R_1}{\longrightarrow}} \left\|\begin{matrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \\ 0 & -2 & k \end{matrix}\right\| \underset{R_4 - R_2}{\overset{R_3 - R_2}{\longrightarrow}} \left\|\begin{matrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & k+1 \end{matrix}\right\|$$

$\overset{R_3 \circlearrowleft R_4}{\longrightarrow} \left\| \begin{matrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & k+1 \\ 0 & 0 & 0 \end{matrix} \right\|$, and from the last matrix it follows that the Rank of $\mathbb{A}$ is equal to 2 iff $k = -1$. To find vectors $\mathbb{X}$ for which $\mathbb{A} \cdot \mathbb{X} = (2, 1, 3, -1)$ we write the condition in matrix form:

$$\left\| \begin{matrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & -2 & -1 \end{matrix} \right\| \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_2 + x_3 = 2 \\ x_1 - x_2 = 1 \\ 2x_1 + x_3 = 3 \\ -2x_2 - x_3 = -1 \end{cases} \Rightarrow$$

$$\begin{cases} x_1 + x_2 + x_3 = 2 \\ -2x_2 - x_3 = -1 \\ -2x_2 - x_3 = -1 \\ -2x_2 - x_3 = -1 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_2 + x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 1 + x_2 \\ x_3 = 1 - 2x_2 \end{cases}$$

; vectors $\mathbb{X}$ are all the vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 + x_2 \\ x_2 \\ 1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

I M 4) The system is an omogeneus one and furthermore it has ∞^2 solutions if and only if $n - \text{Rank}(\mathbb{C}) = 5 - \text{Rank}(\mathbb{C}) = 2$, where n is the number of the unknown variables and $\mathbb{C}$ is the coefficients matrix; to check the Rank we use elementary operations on the lines: $\left\| \begin{matrix} m & n & 1 & 1 & 1 \\ m & n & k & 1 & 1 \\ m & n & k & h & 1 \end{matrix} \right\| \overset{\substack{R_2 - R_1 \\ R_3 - R_2}}{\longrightarrow} \left\| \begin{matrix} m & n & 1 & 1 & 1 \\ 0 & 0 & k-1 & 0 & 0 \\ 0 & 0 & 0 & h-1 & 0 \end{matrix} \right\| \overset{\substack{C_5 \circlearrowleft C_4 \\ C_4 \circlearrowleft C_3}}{\longrightarrow}$

$\left\| \begin{matrix} m & n & 1 & 1 & 1 \\ 0 & 0 & 0 & k-1 & 0 \\ 0 & 0 & 0 & 0 & h-1 \end{matrix} \right\|$ and it is easy to note that Rank($\mathbb{C}$) is 3 iff $h \neq 1$ and $k \neq 1$ independently from m and n.

II M 1) It's easy to verify that on point P the equation is satisfed, the gradient of the functions is $\nabla f(x,y,z) = \left(yz + 2xy + z^2, xz + x^2 - z, xy + 2xz - y\right)$, from which we get $\nabla f(P) = (-2, 1, 2)$; since $f'_z(P) \neq 0$ the equation defines an implicit function z: $(x,y) \to z$ with $\nabla z(1, -1) = \left(-\frac{f'_x(P)}{f'_z(P)}, -\frac{f'_y(P)}{f'_z(P)} \right) = \left(1, -\frac{1}{2}\right)$.

II M 2) $f(x,y) = x^2 + y^2 - x^2 y^2$ and so $\nabla f = \left(2x - 2xy^2, 2y - 2x^2 y\right)$.

$I\,O.C.$: $\begin{cases} f'_x = 2x - 2xy^2 = 0 \\ f'_y = 2y - 2x^2 y = 0 \end{cases} \Rightarrow \begin{cases} 2x(1 - y^2) = 0 \\ 2y(1 - x^2) = 0 \end{cases}$, with five solutions, the point $O = (0,0)$ and the four points $P = ((\pm)1, (\pm)1)$.

$II\,O.C.$: $\mathbb{H}f = \left\| \begin{matrix} 2 - 2y^2 & -4xy \\ -4xy & 2 - 2x^2 \end{matrix} \right\|$,

$|\mathbb{H}f| = \left(2 - 2y^2\right)\left(2 - 2x^2\right) - (-4xy)^2 = 4\left(1 - x^2 - y^2 - 3x^2y^2\right)$.

$\mathbb{H}f(0,0) = \left\| \begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix} \right\| \Rightarrow \begin{cases} f''_{xx}(O) = 2 > 0 \\ |\mathbb{H}f(0,0)| = 4 > 0 \end{cases} \Rightarrow (0,0)$ is a minimum point;

$|\mathbb{H}f(P)| = -16 < 0$ and so the four points P are all saddle points.

II M 3) Problem $\begin{cases} \text{Max/min } f(x,y) = y - x \\ \text{u.c.} \begin{cases} x^2 + y^2 \leq 1 \\ x \leq y \end{cases} \end{cases}$ is equivalent to the problem:

$$\begin{cases} \text{Max/min } f(x,y) = y - x \\ \text{u.c.} \begin{cases} x^2 + y^2 - 1 \le 0 \\ x - y \le 0 \end{cases} \end{cases}$$

; the feasible region is red drawed in the figure in the second next page; the objective function is continuous, the feasible region is bounded and closed, so by Weierstrasse Theorem the problem admits absolute maximum and minimum.

The Lagrangian function is: $\Lambda(x,y,\lambda,\mu) = y - x - \lambda(x^2 + y^2 - 1) - \mu(x - y)$

and its gradient is: $\nabla\Lambda = \begin{pmatrix} -1 - 2\lambda x - \mu \\ 1 - 2\lambda y + \mu \\ -(x^2 + y^2 - 1) \\ -(x - y) \end{pmatrix}$.

KUNH-TUCKER CONDITIONS

First case $(\lambda = 0, \mu = 0)$ (*free optimization*):

$$\begin{cases} \lambda = \mu = 0 \\ -1 = 0 \\ 1 = 0 \\ x^2 + y^2 - 1 \le 0 \\ x - y \le 0 \end{cases}$$

, the system is impossible.

Second case $(\lambda \ne 0, \mu = 0)$ (*first constraint is active*):

$$\begin{cases} \lambda \ne 0,\ \mu = 0 \\ -1 - 2\lambda x = 0 \\ 1 - 2\lambda y = 0 \\ x^2 + y^2 - 1 = 0 \\ x - y \le 0 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0,\ \mu = 0 \\ x = -1/(2\lambda) \\ y = 1/(2\lambda) \\ 1/(4\lambda^2) + 1/(4\lambda^2) = 1 \\ x - y \le 0 \end{cases} \Rightarrow \begin{cases} \lambda \ne 0,\ \mu = 0 \\ x = \mp\sqrt{2}/2 \\ y = \pm\sqrt{2}/2 \\ \lambda = \pm\sqrt{2}/2 \\ \mp\sqrt{2}/2 \mp \sqrt{2}/2 \le 0 \end{cases};$$

if $\lambda = \sqrt{2}/2$, condition $-\sqrt{2}/2 - \sqrt{2}/2 \le 0$ is satisfied; $\left(-\sqrt{2}/2, \sqrt{2}/2\right)$ may be a maximum point $(\lambda > 0)$;

if $\lambda = -\sqrt{2}/2$, condition $\sqrt{2}/2 + \sqrt{2}/2 \le 0$ is not satisfied.

Third case $(\lambda = 0, \mu \ne 0)$ (*second constraint is active*):

$$\begin{cases} \lambda = 0,\ \mu \ne 0 \\ -1 - \mu = 0 \\ 1 + \mu = 0 \\ x^2 + y^2 - 1 \le 0 \\ x - y = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0,\ \mu \ne 0 \\ \mu = -1 \\ \mu = -1 \\ 2x^2 \le 1 \\ y = x \end{cases} \Rightarrow \begin{cases} \lambda = 0,\ \mu \ne 0 \\ \mu = -1 \\ \mu = -1 \\ -\sqrt{2}/2 \le x \le \sqrt{2}/2 \\ y = x \end{cases}$$

, the system has an infinite number of solutions, any point (x,x) such that $-\sqrt{2}/2 \le x \le \sqrt{2}/2$ may be a minimum point $(\mu < 0)$.

Fourth case $(\lambda \ne 0, \mu \ne 0)$ (*both constraints are active*):

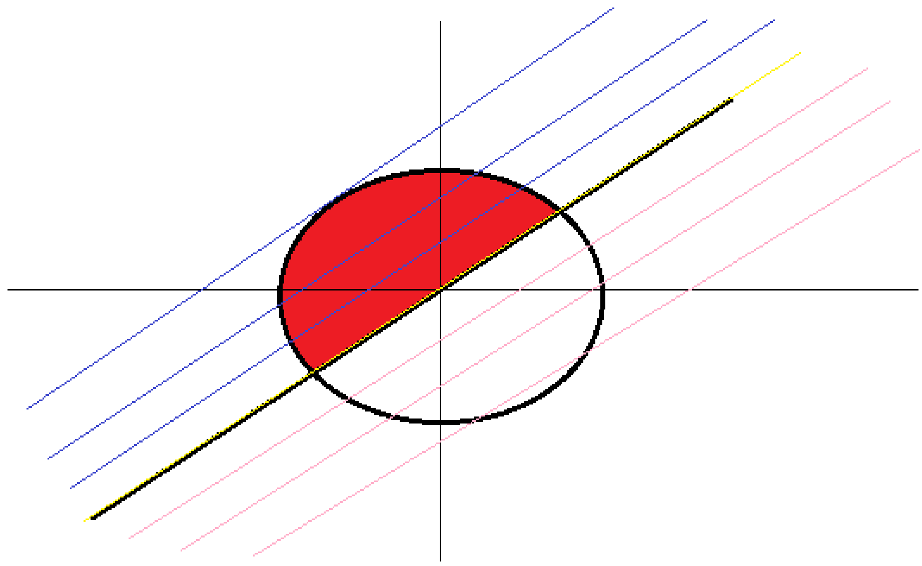
$$\begin{cases} \lambda \neq 0,\ \mu \neq 0 \\ -1 - 2\lambda x - \mu = 0 \\ 1 - 2\lambda y + \mu = 0 \\ x^2 + y^2 - 1 = 0 \\ x - y = 0 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0,\ \mu \neq 0 \\ 2\lambda x + \mu = -1 \\ 2\lambda y - \mu = 1 \\ 2x^2 = 1 \\ y = x \end{cases} \Rightarrow \begin{cases} \lambda \neq 0,\ \mu \neq 0 \\ \pm\sqrt{2}\lambda + \mu = -1 \\ \pm\sqrt{2}\lambda - \mu = 1 \\ x = \pm\sqrt{2}/2 \\ y = \pm\sqrt{2}/2 \end{cases} \Rightarrow$$

$$\begin{cases} \lambda \neq 0,\ \mu \neq 0 \\ \lambda = 0 \\ \mu = -1 \\ x = \pm\sqrt{2}/2 \\ y = \pm\sqrt{2}/2 \end{cases}$$, such solutions cannot be accepted since $\lambda = 0$.

We conclude: $MAX(f) = f\left(-\sqrt{2}/2, \sqrt{2}/2\right) = \sqrt{2}$; $min(f) = f(x,x) = 0$, for $-\sqrt{2}/2 \leq x \leq \sqrt{2}/2$.

For constraints qualification we consider their Jacobian: J $= \begin{Vmatrix} 2x & 2y \\ 1 & -1 \end{Vmatrix}$.

At points $\left(-\sqrt{2}/2, \sqrt{2}/2\right)$ and (x,x), $-\sqrt{2}/2 < x < \sqrt{2}/2$ only one constraint is active and trivia*l*ly qualified, at points (x,x), for $x = \pm\sqrt{2}/2$, the Jacobian's determinant is $\mp 2\sqrt{2} \neq 0$ and constraints are qualified. On the figure below are drawn zero level curve (yellow), positive level curves (blue) and negative level curves (pink).



II M 4) $f(x,y) = xy\,e^{x-y}$ is a twice differentiable function.

$\nabla f = (y\,e^{x-y} + xy\,e^{x-y}, x\,e^{x-y} - xy\,e^{x-y}) = (y\,(1+x)e^{x-y}, x\,(1-y)e^{x-y})$.

$\mathbb{H}f(x,y) = \begin{Vmatrix} y\,e^{x-y} + y\,(1+x)e^{x-y} & (1+x)e^{x-y} - y\,(1+x)e^{x-y} \\ (1-y)e^{x-y} + x\,(1-y)e^{x-y} & -x\,e^{x-y} - x\,(1-y)e^{x-y} \end{Vmatrix} =$

$= \begin{Vmatrix} y\,(2+x)e^{x-y} & (1+x)(1-y)e^{x-y} \\ (1+x)(1-y)e^{x-y} & -x\,(2-y)e^{x-y} \end{Vmatrix}$, $\mathbb{H}f(1,1) = \begin{Vmatrix} 3 & 0 \\ 0 & -1 \end{Vmatrix}$.

$D^2_{v,w}f(1,1) = v \cdot \mathbb{H}f(1,1) \cdot w^T = (\cos\alpha, \sin\alpha) \cdot \begin{Vmatrix} 3 & 0 \\ 0 & -1 \end{Vmatrix} \cdot \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix} =$

$= 3\sin\alpha\cos\alpha - \sin\alpha\cos\alpha = 2\sin\alpha\cos\alpha = \sin(2\alpha)$.

$D^2_{v,w}f(1,1) = 0$ if $\sin(2\alpha) = 0 \Rightarrow (2\alpha = 0)$ or $(2\alpha = \pi) \Rightarrow (\alpha = 0)$ or $\left(\alpha = \frac{\pi}{2}\right)$.