Compito di Analisi Matematica del 7/2/2017 $\boxed{CAM\,1}$

IM1) $z = (i-1)^5 \cdot (i+1)^3 = (\sqrt{2})^5 \cdot \left(\cos\frac{3}{4}\pi + i \,\mathrm{sen}\frac{3}{4}\pi\right)^5 \cdot (\sqrt{2})^3 \cdot \left(\cos\frac{\pi}{4} + i\,\mathrm{sen}\frac{\pi}{4}\right)^3 =$

$= (\sqrt{2})^8 \cdot \left(\cos\frac{15}{4}\pi + i\,\mathrm{sen}\frac{15}{4}\pi\right) \cdot \left(\cos\frac{3}{4}\pi + i\,\mathrm{sen}\frac{3}{4}\pi\right) = 16 \cdot \left(\cos\frac{18}{4}\pi + i\,\mathrm{sen}\frac{18}{4}\pi\right) =$

$= 16 \cdot \left(\cos\frac{9}{2}\pi + i\,\mathrm{sen}\frac{9}{2}\pi\right) = 16 \cdot \left(\cos\frac{\pi}{2} + i\,\mathrm{sen}\frac{\pi}{2}\right) = 16 \cdot i = z.$

$\sqrt[4]{z} = \sqrt[4]{16} \cdot \left(\cos\left(\frac{\pi}{8} + k\cdot\frac{2\pi}{4}\right) + i\,\mathrm{sen}\left(\frac{\pi}{8} + k\cdot\frac{2\pi}{4}\right)\right)\ ;\ 0 \le k \le 3:$

Per $k=0$: $2\cdot\left(\cos\frac{\pi}{8} + i\,\mathrm{sen}\frac{\pi}{8}\right) = 2\cdot\left(\frac{\sqrt{2+\sqrt{2}}}{2} + i\cdot\frac{\sqrt{2-\sqrt{2}}}{2}\right) = \sqrt{2+\sqrt{2}} + i\sqrt{2-\sqrt{2}}.$

Per $k=1$: $2\cdot\left(\cos\frac{5}{8}\pi + i\,\mathrm{sen}\frac{5}{8}\pi\right) = 2\cdot\left(-\frac{\sqrt{2-\sqrt{2}}}{2} + i\frac{\sqrt{2+\sqrt{2}}}{2}\right) = -\sqrt{2-\sqrt{2}} + i\sqrt{2+\sqrt{2}}.$

Per $k=2$: $2\left(\cos\frac{9}{8}\pi + i\,\mathrm{sen}\frac{9}{8}\pi\right) = 2\left(-\frac{\sqrt{2+\sqrt{2}}}{2} - i\frac{\sqrt{2-\sqrt{2}}}{2}\right) = -\sqrt{2+\sqrt{2}} - i\sqrt{2-\sqrt{2}}.$

Per $k=3$: $2\left(\cos\frac{13}{8}\pi + i\,\mathrm{sen}\frac{13}{8}\pi\right) = 2\left(\frac{\sqrt{2-\sqrt{2}}}{2} - i\frac{\sqrt{2+\sqrt{2}}}{2}\right) = \sqrt{2-\sqrt{2}} - i\sqrt{2+\sqrt{2}}.$

IM2) $f(x;y) = \begin{cases} \dfrac{(|x|^\alpha + |y|^\alpha)^3}{\sqrt{x^2+y^2}} & : (x;y) \ne (0;0) \\ 0 & : (x;y) = (0;0) \end{cases}.$

$\lim_{(x;y)\to(0;0)} f(x;y) \Rightarrow \lim_{\rho\to 0} \frac{\rho^{3\alpha}(|\cos^\alpha\vartheta| + |\mathrm{sen}^\alpha\vartheta|)^3}{\rho} = 0$ se $3\alpha - 1 > 0$

$\Rightarrow \alpha > \frac{1}{3}$. La convergenza è uniforme in quanto $(|\cos^\alpha\vartheta| + |\mathrm{sen}^\alpha\vartheta|) \le 2$.

La funzione quindi è continua per $\alpha > \frac{1}{3}$.

$\frac{\partial f}{\partial x}(0;0) = \frac{\partial f}{\partial y}(0;0) = \lim_{h\to 0}\left(\frac{|h|^{3\alpha}}{|h|} - 0\right)\cdot\frac{1}{h} = \lim_{h\to 0} |h|^{3\alpha-2} = 0$ per

$3\alpha - 2 > 0 \Rightarrow \alpha > \frac{2}{3}$. Se $\alpha = \frac{2}{3}$: $\lim_{h\to 0}\frac{|h|}{h} =$ n.e. $\left(\lim_{h\to 0^-} = -1 ; \lim_{h\to 0^+} = 1\right)$

La funzione ammette le derivate parziali in $(0;0)$ se $\alpha > \frac{2}{3}$.

Per vedere se $f(x;y)$ è differenziabile: $\lim_{(x;y)\to(0;0)} \frac{f(x;y) - f(0;0) - \nabla f(0;0)\cdot(x-0; y-0)}{\sqrt{x^2+y^2}} \Rightarrow$

$\Rightarrow \lim_{\rho\to 0}\left[\left(\frac{\rho^{3\alpha}(|\cos^\alpha\vartheta| + |\mathrm{sen}^\alpha\vartheta|)^3}{\rho}\right) - 0 - (0;0)(\rho\cos\vartheta; \rho\,\mathrm{sen}\,\vartheta)\right]\cdot\frac{1}{\rho} \Rightarrow$

CAM2

$\Rightarrow \lim_{\rho \to 0} \rho^{3\alpha-2} \cdot \left(|\cos^\alpha \vartheta| + |\sin^\alpha \vartheta| \right)^3 = 0$ se $3\alpha - 2 > 0 \Rightarrow \alpha > \frac{2}{3}$.

La funzione è differenziabile in $(0;0)$ per $\alpha > \frac{2}{3}$.

IM3) $f(x;y) = e^{y^2-x^2} - e^{x-y}$. $f(P) = f(1;1) = e^0 - e^0 = 0$.

$\nabla f(x;y) = \left(-2x e^{y^2-x^2} - e^{x-y} ; 2y e^{y^2-x^2} + e^{x-y}\right)$. $\nabla f(1;1) = (-3;3)$.

$f''_{xx} = -2e^{y^2-x^2} + 4x^2 e^{y^2-x^2} - e^{x-y}$; $f''_{xx}(1;1) = 1$.

$f''_{xy} = -4xy\, e^{y^2-x^2} + e^{x-y}$; $f''_{xy}(1;1) = -3$.

$f''_{yy} = 2e^{y^2-x^2} + 4y^2 e^{y^2-x^2} - e^{x-y}$; $f''_{yy}(1;1) = 5$.

$y'(1) = -\frac{-3}{3} = 1$; $y''(1) = -\frac{1 + 2(-3)\cdot 1 + 5\cdot(1)^2}{3} = -\frac{1-6+5}{3} = 0$.

$P_2(x;1) = 1 + 1(x-1) + \frac{0}{2}(x-1)^2 = x$.

IM4) $f(x;y) = x^2y + xy^2$. $v = \left(\frac{1}{\sqrt{2}} ; \frac{1}{\sqrt{2}}\right)$; $w = \left(-\frac{1}{\sqrt{2}} ; \frac{1}{\sqrt{2}}\right)$

$\nabla f(x;y) = (2xy + y^2 ; x^2 + 2xy)$; $\nabla f(1;-1) = (-1;-1)$.

$\mathcal{D}_v f(1;-1) = \nabla f(1;-1) \bullet v = (-1;-1)\cdot\left(\frac{1}{\sqrt{2}} ; \frac{1}{\sqrt{2}}\right) = -\frac{2}{\sqrt{2}} = -\sqrt{2}$.

$H(x;y) = \begin{Vmatrix} 2y & 2x+2y \\ 2x+2y & 2x \end{Vmatrix}$; $H(1;-1) = \begin{Vmatrix} -2 & 0 \\ 0 & 2 \end{Vmatrix}$.

$\mathcal{D}^2_{v,w} f(1;-1) = \begin{Vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} -2 & 0 \\ 0 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{Vmatrix} = \begin{Vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} \sqrt{2} \\ \sqrt{2} \end{Vmatrix} = 1 + 1 = 2$.

IIM1) $\begin{cases} \text{Max/min } f(x;y) = x^3 + y^3 \\ \text{s.v. } x^2 + y^2 \leq 4 \end{cases}$. Σ insieme limitato e chiuso $f(x;y)$ funzione continua, vincolo qualificato.

$\Lambda = x^3 + y^3 - \lambda(x^2 + y^2 - 4)$.

Caso $\lambda = 0$: $\begin{cases} \Lambda'_x = 3x^2 = 0 \\ \Lambda'_y = 3y^2 = 0 \\ x^2 + y^2 \leq 4 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \leq 4 \end{cases}$; $H(x;y) = \begin{Vmatrix} 6x & 0 \\ 0 & 6y \end{Vmatrix}$. $H(0;0) = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}$??

CAM3

$f(0;0)=0$; $f(x;0)=x^3>0$ per $x>0$; $x^3<0$ per $x<0$.

Quindi $(0;0)$ è un punto di Sella.

<u>Caso $\lambda \neq 0$</u> $\begin{cases} \Lambda'_x = 3x^2-2\lambda x = x(3x-2\lambda)=0 \\ \Lambda'_y = 3y^2-2\lambda y = y(3y-2\lambda)=0 \\ x^2+y^2=4 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ 0=4 \\ \text{imp.} \end{cases} \cup \begin{cases} x=0 \\ \lambda=\frac{3}{2}y \\ y^2=4 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=2 \\ \lambda=3 \\ \text{Max?} \end{cases} \cup \begin{cases} x=0 \\ y=-2 \\ \lambda=-3 \\ \text{min?} \end{cases}$

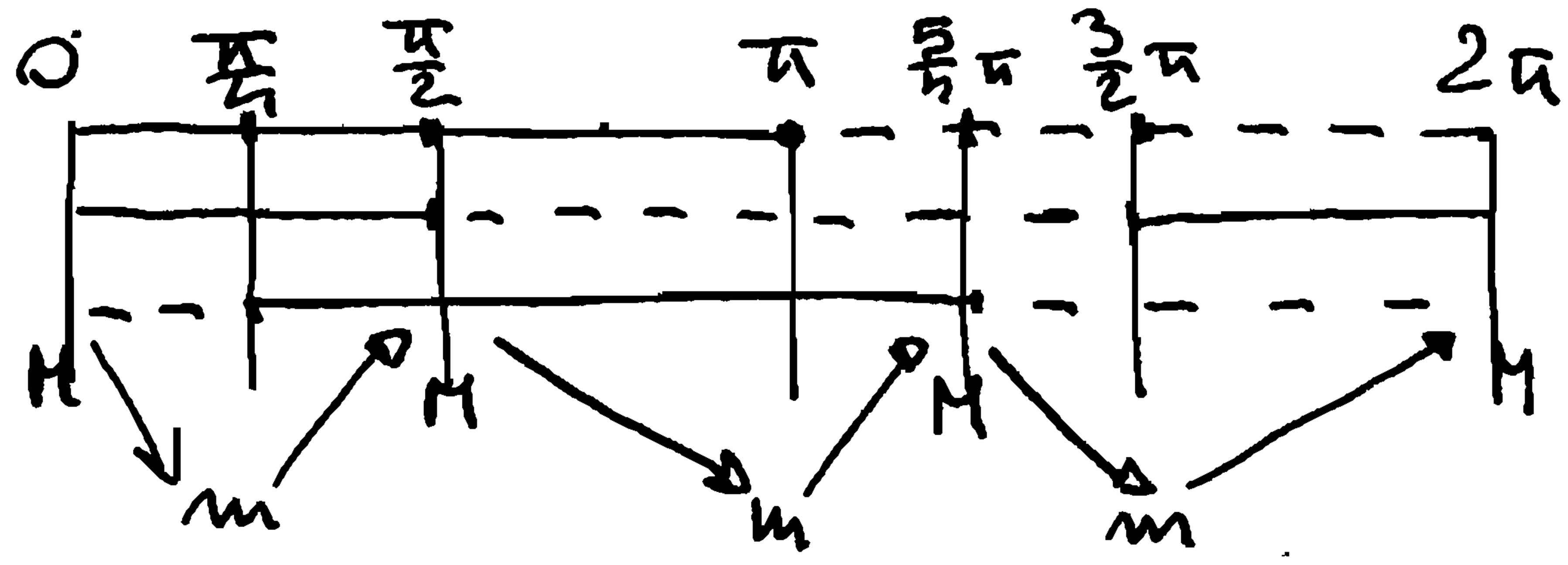
$\Rightarrow \begin{cases} \lambda=\frac{3}{2}x \\ y=0 \\ x^2=4 \end{cases} \Rightarrow \begin{cases} x=2 \\ y=0 \\ \lambda=3 \\ \text{Max?} \end{cases} \cup \begin{cases} x=-2 \\ y=0 \\ \lambda=-3 \\ \text{min?} \end{cases} \cup \begin{cases} x=\frac{2}{3}\lambda \\ y=\frac{2}{3}\lambda \\ \frac{4}{9}\lambda^2+\frac{4}{9}\lambda^2=4 \end{cases} \Rightarrow \lambda^2=\frac{9}{2} \Rightarrow \lambda=\pm\frac{3}{\sqrt{2}} \Rightarrow$

$\Rightarrow \begin{cases} x=\sqrt{2} \\ y=\sqrt{2} \\ \lambda=\frac{3}{\sqrt{2}} \\ \text{Max?} \end{cases} \cup \begin{cases} x=-\sqrt{2} \\ y=-\sqrt{2} \\ \lambda=-\frac{3}{\sqrt{2}} \\ \text{min?} \end{cases}$. Studiamo la funzione sulla frontiera ponendo $x=2\cos t$; $y=2\operatorname{sen} t$.

$f(x;y)=x^3+y^3 \Rightarrow f(t)=8\cos^3 t+8\operatorname{sen}^3 t$.

$f'(t)=24\cos^2 t(-\operatorname{sen} t)+24\operatorname{sen}^2 t\cdot\cos t=24\operatorname{sen} t\cos t\cdot(\operatorname{sen} t-\cos t)\geq 0$

$\operatorname{sen} t\geq 0$: $0\leq x\leq\pi$
$\cos t\geq 0$: $0\leq x\leq\frac{\pi}{2}\cup\frac{3}{2}\pi\leq x\leq 2\pi$
$\operatorname{sen} t\geq\cos t$: $\frac{\pi}{4}\leq x\leq\frac{5}{4}\pi$



<u>Conclusioni</u>:

$(0;2)$ è un punto di Massimo ; $(2;0)$ è un punto di Massimo ;
$(0;-2)$ è un punto di minimo ; $(-2;0)$ è un punto di minimo.
$(\sqrt{2};\sqrt{2})$ $(t=\frac{\pi}{4})$ non è un punto di Massimo ; $(-\sqrt{2};-\sqrt{2})$ $(t=\frac{5}{4}\pi)$ non è un punto di minimo.

II M2) $\begin{cases} y'\cdot\log^2 y=2xy \\ y(0)=1 \end{cases} \Rightarrow \frac{1}{y}\cdot\log^2 y\cdot y'=2x$ con $y>0 \Rightarrow$

$\Rightarrow \int\frac{1}{y}\cdot\log^2 y\,dy=\int 2x\,dx+K \Rightarrow \int\log^2 y\,d(\log y)=x^2+K \Rightarrow \frac{1}{3}\log^3 y=x^2+K \Rightarrow$

$\Rightarrow \log^3 y=3x^2+m \Rightarrow \log y=\sqrt[3]{3x^2+m} \Rightarrow y=e^{\sqrt[3]{3x^2+m}}$.

$\boxed{CAM4}$

$y(0)=1 \Rightarrow 1 = e^{\sqrt[3]{m}} \Rightarrow m = 0 \Rightarrow y = e^{\sqrt[3]{3x^2}}$.

II M3) $\begin{cases} x' = x+2y \\ y' = -x+3y \end{cases} \Rightarrow \begin{cases} x'-x-2y=0 \\ x+y'-3y=0 \end{cases} \Rightarrow \begin{cases} (D-1)(x)-2y=0 \\ x+(D-3)y=0 \end{cases}$.

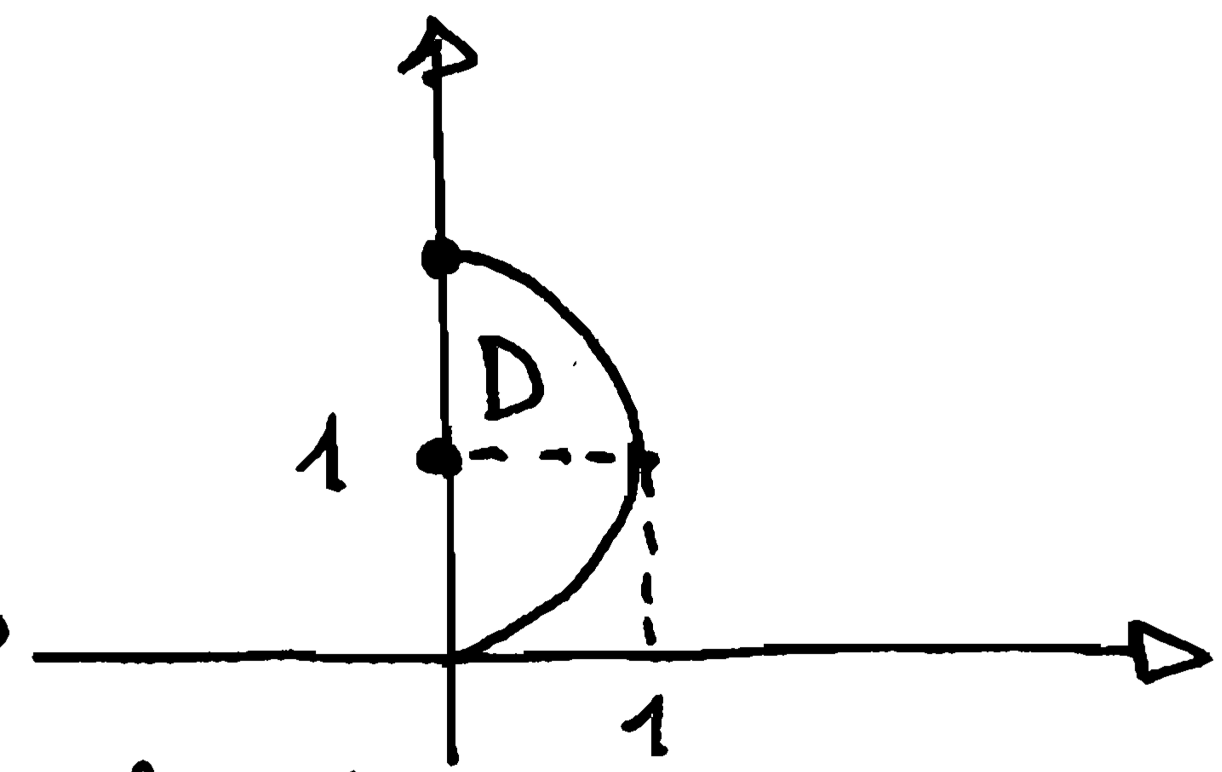
$\begin{vmatrix} D-1 & -2 \\ 1 & D-3 \end{vmatrix}(x) = (D^2-4D+5)(x)=0 \Rightarrow x''-4x'+5x=0.$

$\lambda^2-4\lambda+5=0 \Rightarrow \lambda = 2 \pm \sqrt{4-5} = 2 \pm i.$

Soluzione $x(t) = c_1 e^{2t} \operatorname{sen} t + c_2 e^{2t} \cos t$. Dalla 1ª Equazione:

$y(t) = \frac{1}{2}(x'-x) = \frac{1}{2}\left(2c_1 e^{2t}\operatorname{sen} t + c_1 e^{2t}\cos t + 2c_2 e^{2t}\cos t - c_2 e^{2t}\operatorname{sen} t - c_1 e^{2t}\operatorname{sen} t + \right.$
$\left. - c_2 e^{2t}\cos t\right) = \frac{1}{2}\left(c_1 e^{2t}\operatorname{sen} t + c_1 e^{2t}\cos t + c_2 e^{2t}\cos t - c_2 e^{2t}\operatorname{sen} t\right) =$
$= y(t) = \frac{1}{2}c_1 e^{2t}(\operatorname{sen} t + \cos t) + \frac{1}{2}c_2 e^{2t}(\cos t - \operatorname{sen} t).$

II M4) $\iint_D xy^2\,dx\,dy$; $D = \{(x;y): 0 \le x;\ x^2+y^2 \le 2y\}$.



$x^2+y^2-2y \le 0 \Rightarrow x^2+y^2-2y+1 \le 1 \Rightarrow (x-0)^2+(y-1)^2 \le 1^2.$

Se $x = \rho\cos\vartheta$ e $y = \rho\operatorname{sen}\vartheta \Rightarrow \rho^2\cos^2\vartheta + \rho^2\operatorname{sen}^2\vartheta \le 2\rho\operatorname{sen}\vartheta \Rightarrow$
$\Rightarrow \rho^2 - 2\rho\operatorname{sen}\vartheta = 0 \Rightarrow \rho(\rho - 2\operatorname{sen}\vartheta) = 0 \Rightarrow \rho = 2\operatorname{sen}\vartheta$, che rappresenta l'equazione della circonferenza $x^2+y^2=2y$ in coordinate polari.

$$\iint_D x\cdot y^2\,dx\,dy = \int_0^{\frac{\pi}{2}}\int_0^{2\operatorname{sen}\vartheta}(\rho\cos\vartheta\cdot\rho^2\operatorname{sen}^2\vartheta)\cdot\rho\,d\rho\,d\vartheta =$$
$$= \int_0^{\frac{\pi}{2}}\left(\int_0^{2\operatorname{sen}\vartheta}\rho^4\,d\rho\right)\cdot\cos\vartheta\cdot\operatorname{sen}^2\vartheta\,d\vartheta = \int_0^{\frac{\pi}{2}}\left(\frac{1}{5}\rho^5\Big|_0^{2\operatorname{sen}\vartheta}\right)\cdot\cos\vartheta\cdot\operatorname{sen}^2\vartheta\,d\vartheta =$$
$$= \int_0^{\frac{\pi}{2}}\frac{1}{5}\cdot 32\cdot\operatorname{sen}^7\vartheta\cdot\cos\vartheta\,d\vartheta = \frac{32}{5}\int_0^{\frac{\pi}{2}}\operatorname{sen}^7\vartheta\,d(\operatorname{sen}\vartheta) = \frac{32}{5}\cdot\left(\frac{1}{8}\operatorname{sen}^8\vartheta\right)\Big|_0^{\frac{\pi}{2}} =$$
$$= \frac{4}{5}(1-0) = \frac{4}{5}.$$