TASK MATHEMATICS for ECONOMIC APPLICATIONS
07/02/2017

I M 1) $\sqrt[3]{(i^2-1)^5(i+1)^2}=\sqrt[3]{(-1-1)^5(i^2+2i+1)}=$
$=\sqrt[3]{(-2)^5(-1+2i+1)}=\sqrt[3]{(-2)^5\cdot 2i}=\sqrt[3]{-2^6 i}=2^2\sqrt[3]{-i}=$
$=4\sqrt[3]{\cos\left(\frac{3}{2}\pi\right)+i\sin\left(\frac{3}{2}\pi\right)}=4\left(\cos\left(\frac{\pi}{2}+\frac{2}{3}k\pi\right)+i\sin\left(\frac{\pi}{2}+\frac{2}{3}k\pi\right)\right),$
$0\le k\le 2$. The three roots are:
$x_1=4\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)=4i\,;$
$x_2=4\left(\cos\left(\frac{7}{6}\pi\right)+i\sin\left(\frac{7}{6}\pi\right)\right)=4\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=-2\left(\sqrt{3}+i\right);$
$x_3=4\left(\cos\left(\frac{11}{6}\pi\right)+i\sin\left(\frac{11}{6}\pi\right)\right)=4\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=2\left(\sqrt{3}-i\right).$

I M 2) The characteristic polynomial of $\mathbb{A}$ is

$$p_{\mathbb{A}}(\lambda)=|\mathbb{A}-\lambda\mathbb{I}|=\begin{vmatrix}3-\lambda & 2 & 1\\ 1 & 4-\lambda & k\\ 1 & 2 & -3-\lambda\end{vmatrix}=$$
$$=(3-\lambda)\begin{vmatrix}4-\lambda & k\\ 2 & -3-\lambda\end{vmatrix}-2\begin{vmatrix}1 & k\\ 1 & -3-\lambda\end{vmatrix}+\begin{vmatrix}1 & 4-\lambda\\ 1 & 2\end{vmatrix}=$$
$$=(3-\lambda)\left(\lambda^2-\lambda-12-2k\right)-2(-3-\lambda-k)+(2-4+\lambda)=$$
$$=-\lambda^3+4\lambda^2+2(6+k)\lambda-4(8+k)=(\lambda-2)\left(-\lambda^2+2\lambda+2(8+k)\right).$$

If we define $q(\lambda)=-\lambda^2+2\lambda+2(8+k)$, $\lambda=2$ is a multiple eingenvalue for $\mathbb{A}$ iff $q(2)=2(8+k)=0$ or $k=-8$; in such case the characteristic polynomial $p_{\mathbb{A}}(\lambda)$ can be factorized as $p_{\mathbb{A}}(\lambda)=-\lambda(\lambda-2)^2$ and the three eingevalues are $\lambda_{1,2}=2$ and $\lambda_3=0$. To check if $\mathbb{A}$ is a diagonalizable matrix we need the geometric multiplicity of $\lambda=2$ to be equal to 2, its algebraic multiplicity.

For this goal we consider a generic element $\mathbb{X}=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}\in\mathcal{ES}_{\lambda=2}$, $\mathbb{X}$ must satisfy the condition i.e. the linear system:

$$\begin{cases}x_1+2x_2+x_3=0\\ x_1+2x_2-8x_3=0\\ x_1+2x_2-5x_3=0\end{cases}\Rightarrow\begin{cases}x_1+2x_2+x_3=0\\ -9x_3=0\\ -6x_3=0\end{cases}\Rightarrow\begin{cases}x_1=-2x_2\\ x_3=0\end{cases}$$

. Every element of $\mathcal{ES}_{\lambda=2}$ is a vector $\begin{pmatrix}-2x_2\\ x_2\\ 0\end{pmatrix}=x_2\begin{pmatrix}-2\\ 1\\ 0\end{pmatrix}$; the geometric multiplicity of $\lambda=2$ is only 1 and the matrix is not diagonalizable.

I M 3) The dim(Imm) of a linear map is equal to the rank of the matrix $\mathbb{A}$ associated at the map; so by elementary operations on matrix's lines we get:

$$\left\|\begin{matrix}1 & 1 & 2 & -2\\ 1 & 2 & 1 & 1\\ 3 & 2 & m & k\end{matrix}\right\| \overset{\substack{R_2-R_1\\ R_3-3R_1}}{\longrightarrow} \left\|\begin{matrix}1 & 1 & 2 & -2\\ 0 & 1 & -1 & 3\\ 0 & -1 & m-6 & k+6\end{matrix}\right\| \overset{R_3+R_2}{\longrightarrow}$$

$\rightarrow \left\|\begin{matrix}1 & 1 & 2 & -2\\ 0 & 1 & -1 & 3\\ 0 & 0 & m-7 & k+9\end{matrix}\right\|$, and so we get:

$\text{Rank}(\mathbb{A}) = \begin{cases} 2 & \text{if } m=7 \wedge k=-9 \\ 3 & \text{otherwise}\end{cases}$. By the dimention Theorem:

$\dim(\text{Ker}) + \dim(\text{Imm}) = \dim(\mathbb{R}^4) = 4$ and trivially dim(Ker) is maximum if and only if dim(Imm) is minimum and so $m = 7 \wedge k = -9$ with $\dim(\text{Imm}) = \dim(\text{Ker}) = 2$.

To find a basis for Ker remember that $\mathbb{X}$ belongs to the kernel of f if $\mathbb{A}\cdot\mathbb{X} = \mathbb{O}$, i.e.:

$$\begin{cases} x_1 + x_2 + 2x_3 - 2x_4 = 0 \\ x_1 + 2x_2 + x_3 + x_4 = 0 \\ 3x_1 + 2x_2 + 7x_3 - 9x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 - 2x_4 = 0 \\ x_2 - x_3 + 3x_4 = 0 \\ -x_2 + x_3 - 3x_4 = 0 \end{cases} \Rightarrow$$

$\Rightarrow \begin{cases} x_1 = -3x_3 + 5x_4 \\ x_2 = x_3 - 3x_4 \end{cases}$. Every element of the Kernel is a vector:

$\mathbb{X} = (-3x_3 + 5x_4, x_3 - 3x_4, x_3, x_4) = x_3(-3,1,1,0) + x_4(5,-3,0,1)$ and a basis for the Kernel is $\mathcal{B}_{Ker(f)} = \{(-3,1,1,0),(5,-3,0,1)\}$.

To find a basis for Imm note that a vector $\mathbb{Y}$ belongs to the image of f if $\mathbb{A}\cdot\mathbb{X} = \mathbb{Y}$, that in system form is:

$$\begin{cases} x_1 + x_2 + 2x_3 - 2x_4 = y_1 \\ x_1 + 2x_2 + x_3 + x_4 = y_2 \\ 3x_1 + 2x_2 + 7x_3 - 9x_4 = y_3 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 - 2x_4 = y_1 \\ x_2 - x_3 + 3x_4 = y_2 - y_1 \\ -x_2 + x_3 - 3x_4 = y_3 - 3y_1 \end{cases} \Rightarrow$$

$\Rightarrow \begin{cases} x_1 + x_2 + 2x_3 - 2x_4 = y_1 \\ x_2 - x_3 + 3x_4 = y_2 - y_1 \\ 0 = y_3 + y_2 - 4y_1 \end{cases}$. From $y_3 + y_2 - 4y_1 = 0$ we get $y_3 = 4y_1 - y_2$ and so every element of Imm is a vector:

$\mathbb{Y} = (y_1, y_2, y_3) = (y_1, y_2, 4y_1 - y_2) = y_1(1,0,4) + y_2(0,1,-1)$ and a basis for the Image is $\mathcal{B}_{Imm(f)} = \{(1,0,4),(0,1,-1)\}$.

I M 4) If two matrices are similar they have the same characteristic polynomial, and so, calculating the two polynomials we get:

$p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 1-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 1$ and

$p_{\mathbb{B}}(\lambda) = |\mathbb{B} - \lambda\mathbb{I}| = \begin{vmatrix} -\lambda & -1 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 1$, so $p_{\mathbb{A}}(\lambda) = p_{\mathbb{B}}(\lambda)$.

For similarity we need a non singular matrix $\mathbb{P}$ such that:

$\mathbb{A}\cdot\mathbb{P} = \mathbb{P}\cdot\mathbb{B} \Rightarrow \left\|\begin{matrix}1 & 2\\ 1 & 3\end{matrix}\right\| \cdot \left\|\begin{matrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{matrix}\right\| = \left\|\begin{matrix}p_{11} & p_{12}\\ p_{21} & p_{22}\end{matrix}\right\| \cdot \left\|\begin{matrix}0 & -1\\ 1 & 4\end{matrix}\right\| \Rightarrow$

$\Rightarrow \left\|\begin{matrix}p_{11}+2p_{21} & p_{12}+2p_{22}\\ p_{11}+3p_{21} & p_{12}+3p_{22}\end{matrix}\right\| = \left\|\begin{matrix}p_{12} & 4p_{12}-p_{11}\\ p_{22} & 4p_{22}-p_{21}\end{matrix}\right\|$ and in system form we get:

$$\begin{cases} p_{11} + 2p_{21} = p_{12} \\ p_{12} + 2p_{22} = 4p_{12} - p_{11} \\ p_{11} + 3p_{21} = p_{22} \\ p_{12} + 3p_{22} = 4p_{22} - p_{21} \end{cases} \Rightarrow \begin{cases} p_{11} + 2p_{21} = p_{12} \\ p_{11} + 2p_{22} = 3p_{12} \\ p_{11} + 3p_{21} = p_{22} \\ p_{12} + p_{21} = p_{22} \end{cases} \Rightarrow \begin{cases} p_{11} = 2p_{21} - p_{12} \\ p_{22} = p_{12} + p_{21} \end{cases}.$$

$\mathbb{P}$ is any matrix of the form $\mathbb{P} = \left\|\begin{matrix}2p-q & q\\ p & p+q\end{matrix}\right\|$ with $q \neq \pm\sqrt{2}p$.

II M 1) It is easy to verify that on point P the given equation is satisfied.
The gradient of the functions is $\nabla f = \Big(-2xe^{y^2-x^2} - e^{x-y}, 2ye^{y^2-x^2} + e^{x-y} \Big)$ from which $\nabla f(P) = (-3, 3)$; since $f'_y(P) \neq 0$ the equation defines an implicit function $x \to y(x)$ with $y'(1) = -\dfrac{f'_x(P)}{f'_y(P)} = -\dfrac{-3}{3} = 1$. The Hessian matrix is:

$$\mathbb{H}(x,y) = \begin{Vmatrix} 2(2x^2-1)e^{y^2-x^2} - e^{x-y} & -4xye^{y^2-x^2} + e^{x-y} \\ -4xye^{y^2-x^2} + e^{x-y} & 2(2y^2+1)e^{y^2-x^2} - e^{x-y} \end{Vmatrix}$$, from which:

$$\mathbb{H}(1,1) = \begin{Vmatrix} 1 & -3 \\ -3 & 5 \end{Vmatrix}.$$

Since $y''(1) = -\dfrac{f''_{xx}(P) + 2\,f''_{xy}(P)\cdot y'(1) + f''_{yy}(P)\cdot (y'(1))^2}{f'_y(P)}$ we get

$$y''(1) = -\frac{1 + 2(-3)\cdot 1 + 5\cdot 1}{3} = 0.$$

II M 2) The Lagrangian function of the problem is
$\Lambda(x,y,z,\lambda) = x^2 + y^2 + z^2 - \lambda(x+y+z+1)$ and its gradient is:
$\nabla\Lambda = (2x-\lambda, 2y-\lambda, 2z-\lambda, -(x+y+z+1))$.

$$I\,O.C.: \begin{cases} 2x - \lambda = 0 \\ 2y - \lambda = 0 \\ 2z - \lambda = 0 \\ x+y+z = -1 \end{cases} \Rightarrow \begin{cases} x = y = z = \lambda/2 \\ 3\lambda/2 = -1 \end{cases}$$, with only one solution, the point $P = (-1/3, -1/3, -1/3, -2/3)$.

$$II\,O.C.: \overline{\mathbb{H}}(\Lambda) = \begin{Vmatrix} 0 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{Vmatrix}$$, since the problem has three variables and one constraint, we must consider two principal minors, $\overline{\mathbb{H}}_3(\Lambda)$ and $\overline{\mathbb{H}}_4(\Lambda)$.

$$\overline{\mathbb{H}}_3(P) = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix} = -2 - 2 = -4 < 0;$$

$$\overline{\mathbb{H}}_4(P) = \begin{vmatrix} 0 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{vmatrix} =$$

$$= \begin{vmatrix} -1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix} - \begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 2 \end{vmatrix} + + \begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 2 \\ -1 & 0 & 0 \end{vmatrix} = -4 - 4 - 4 = -12 < 0.$$

On point P, $\overline{\mathbb{H}}_3(P)$ and $\overline{\mathbb{H}}_4(P)$ are both negative, so P is a minimum point with $\min(f) = f(-1/3, -1/3, -1/3) = 1/3$.

II M 3) The function $f(x,y) = x^2y + xy^2$ is a twice differentiable function, so $D_v f(P_0) = \nabla f(P_0)\cdot v$ and $D^2_{v,w} f(P_0) = v \cdot \nabla f(P_0) \cdot w^{\mathrm{T}}$.

Since $v = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $w = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\nabla f(x,y) = \left(2xy + y^2, x^2 + 2xy\right)$,
$\nabla f(P_0) = (-1, -1)$; $\mathbb{H}(x,y) = \left\| \begin{matrix} 2y & 2(x+y) \\ 2(x+y) & 2x \end{matrix} \right\|$, $\mathbb{H}(P_0) = \left\| \begin{matrix} -2 & 0 \\ 0 & 2 \end{matrix} \right\|$.
And so:

$$D_v f(P_0) = \nabla f(P_0) \cdot v = (-1, -1) \cdot \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) = -\sqrt{2};$$

$$D^2_{v,w} f(P_0) = v \cdot \nabla f(P_0) \cdot w^{\mathrm{T}} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \left\| \begin{matrix} -2 & 0 \\ 0 & 2 \end{matrix} \right\| \cdot \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} =$$

$$= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \cdot \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \end{pmatrix} = 2\,.$$

II M 4) $\nabla f = \begin{pmatrix} f'_x \\ f'_y \\ f'_z \end{pmatrix} = \begin{pmatrix} (2y - z^3)x^{2y-z^3-1} + (y+2)^3 (xz)^{(y+2)^3-1} \cdot z \\ x^{2y-z^3} \cdot log\, x \cdot 2 + (xz)^{(y+2)^3} \cdot log(xz) \cdot 3(y+2)^2 \\ x^{2y-z^3} \cdot log\, x \cdot (-3z^2) + (y+2)^3 (xz)^{(y+2)^3-1} \cdot x \end{pmatrix}$.