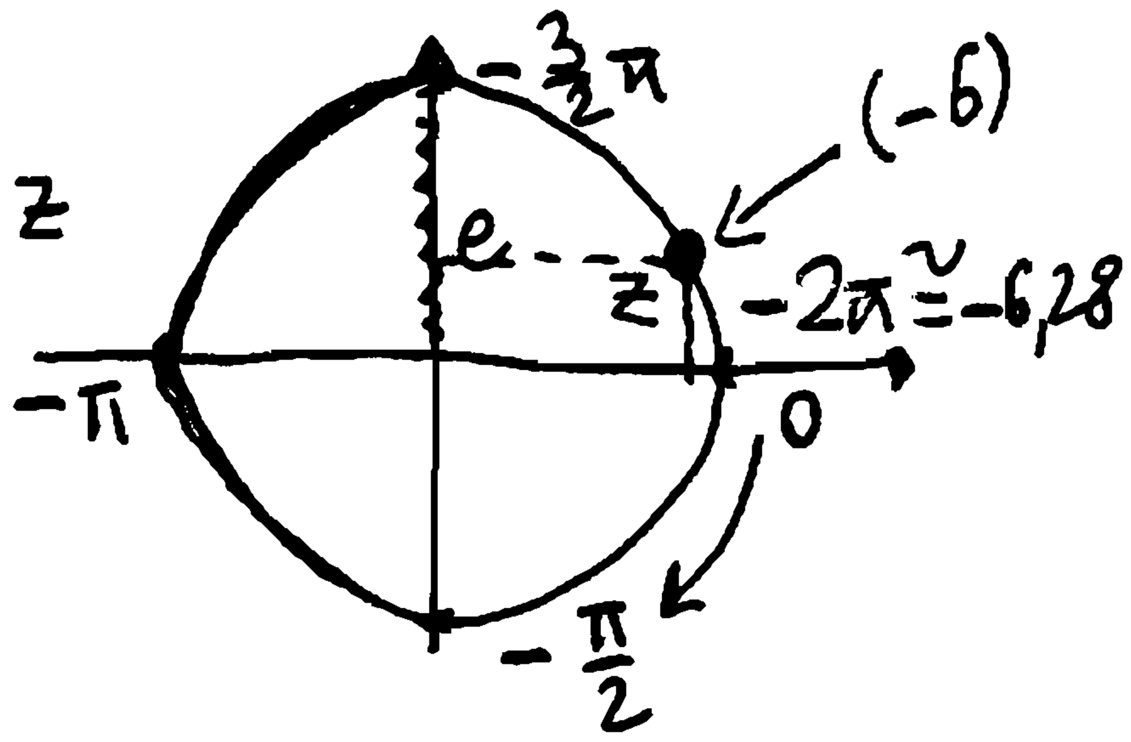
Compito di Analisi Matematica del 16/3/2017 $\boxed{\text{AM1}}$



IM1) $e^{1-6i} = e^1 \cdot (\cos(-6) + i\,\text{sen}(-6)) = (e\cdot\cos(-6)) + i\,(e\cdot\text{sen}(-6)) = z$

IM2) $f(x;y) = \begin{cases} \dfrac{xy(x+y)}{x^2+y^2} & : (x;y) \neq (0;0) \\ 0 & : (x;y) = (0;0) \end{cases}$.

$$\lim_{(x;y)\to(0;0)} f(x;y) \Rightarrow \lim_{\rho\to 0} \frac{\rho^3\cdot\cos\vartheta\cdot\text{sen}\vartheta\cdot(\cos\vartheta+\text{sen}\vartheta)}{\rho^2} = \lim_{\rho\to 0}\rho\cdot\left(\cos\vartheta\,\text{sen}\vartheta(\cos\vartheta+\text{sen}\vartheta)\right) = 0$$

La convergenza è uniforme in quanto $\left|\rho\cdot\cos\vartheta\cdot\text{sen}\vartheta\cdot(\cos\vartheta+\text{sen}\vartheta)\right| < \rho\cdot 1\cdot 1\cdot 2 = 2\rho$.

Quindi $f(x;y)$ è continua in $(0;0)$.

$$\frac{\partial f}{\partial x}(0;0) = \lim_{h\to 0}\left(\frac{(0+h)\cdot 0(0+h+0)}{h^2+0^2}\right)\cdot\frac{1}{h} = \frac{\partial f}{\partial y}(0;0) = 0 \Rightarrow \nabla f(0;0) = (0;0).$$

$$\lim_{(x;y)\to(0;0)} \frac{f(x;y) - f(0;0) - \nabla f(0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}} = \lim_{(x;y)\to(0;0)} \frac{xy(x+y)}{(x^2+y^2)\sqrt{x^2+y^2}} \Rightarrow$$

$$\Rightarrow \lim_{\rho\to 0}\frac{\cancel{\rho^3}(\cos\vartheta\cdot\text{sen}\vartheta\cdot(\cos\vartheta+\text{sen}\vartheta))}{\cancel{\rho^2\cdot\rho}} = \cos\vartheta\cdot\text{sen}\vartheta\cdot(\cos\vartheta+\text{sen}\vartheta) \neq 0.$$

La funzione non è differenziabile in $(0;0)$.

IM3) $f(x;y;z) = xe^{y+z} - ye^{x+z} + ze^{x+y} = 0$. $f(1;1;0) = 1\cdot e - 1\cdot e - 0 = 0$.

$\nabla f(x;y;z) = \left(e^{y+z} - ye^{x+z} + ze^{x+y};\ xe^{y+z} - e^{x+z} + z\cdot e^{x+y};\ xe^{y+z} - ye^{x+z} + e^{x+y}\right)$.

$\nabla f(1;1;0) = (e - e + 0;\ e - e + 0;\ e - e + e^2) = (0;0;e^2)$. Esiste $(x;y)\to z(x;y)$.

$$\frac{\partial z}{\partial z} = -\frac{f'_x}{f'_z} = -\frac{0}{e^2} = 0;\ \frac{\partial z}{\partial y} = -\frac{f'_y}{f'_z} = -\frac{0}{e^2} = 0 \Rightarrow \nabla z(1;1) = (0;0).$$

IM4) $f(x;y) = (x+y)\cdot e^{(x-y)}$

$$P_2((x;y);(0;0)) = f(0;0) + \nabla f(0;0)\cdot(x-0;y-0) + \frac{1}{2}(x-0;y-0)\cdot H(0;0)\cdot(x-0;y-0)^T \Rightarrow$$

$$\Rightarrow \nabla f(x;y) = \left(e^{(x-y)} + (x+y)\cdot e^{(x-y)};\ e^{(x-y)} - (x+y)e^{(x-y)}\right) = \left((1+x+y)e^{x-y};\ (1-x-y)e^{x-y}\right).$$

AM2

$\Rightarrow \nabla f(0;0) = (1;1)$.

$$H(x;y) = \begin{Vmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{Vmatrix} = \begin{Vmatrix} 1\cdot e^{x-y} + (1+x+y)\,e^{x-y} & 1\cdot e^{x-y} - (1+x+y)\,e^{x-y} \\ -1\cdot e^{x-y} + (1-x-y)\,e^{x-y} & -1\, e^{x-y} - (1-x-y)\,e^{x-y} \end{Vmatrix} \Rightarrow$$

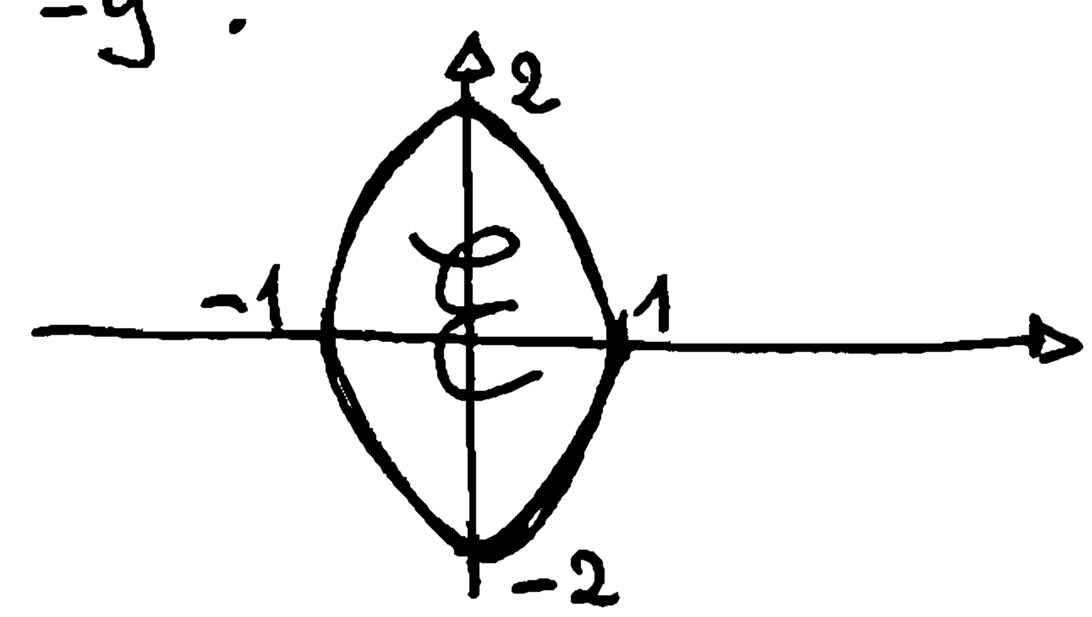
$$\Rightarrow H(x;y) = \begin{Vmatrix} (2+x+y)\,e^{x-y} & -(x+y)\,e^{x-y} \\ -(x+y)\,e^{x-y} & (x+y-2)\,e^{x-y} \end{Vmatrix} \Rightarrow H(0;0) = \begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix}.$$

$$P_2((x;y);(0;0)) = f(0;0) + \nabla f(0;0)\cdot(x-0;y-0) + \tfrac{1}{2}(x-0;y-0)\cdot H(0;0)\cdot(x-0;y-0)^T =$$

$$P_2((x;y);(0;0)) = 0 + (1;1)\cdot(x;y) + \tfrac{1}{2}\begin{Vmatrix} x & y \end{Vmatrix}\cdot\begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix}\cdot\begin{Vmatrix} x \\ y \end{Vmatrix} = x+y+\tfrac{1}{2}\cdot\begin{Vmatrix} x & y \end{Vmatrix}\cdot\begin{Vmatrix} 2x \\ -2y \end{Vmatrix} =$$

$$P_2((x;y);(0;0)) = x+y+\tfrac{1}{2}(2x^2-2y^2) = x+y+x^2-y^2.$$

II M1) $\begin{cases} \text{Max/min } f(x;y) = x^2+xy^2 \\ \text{s.v. } 4x^2+y^2 \leq 4 \end{cases}$.



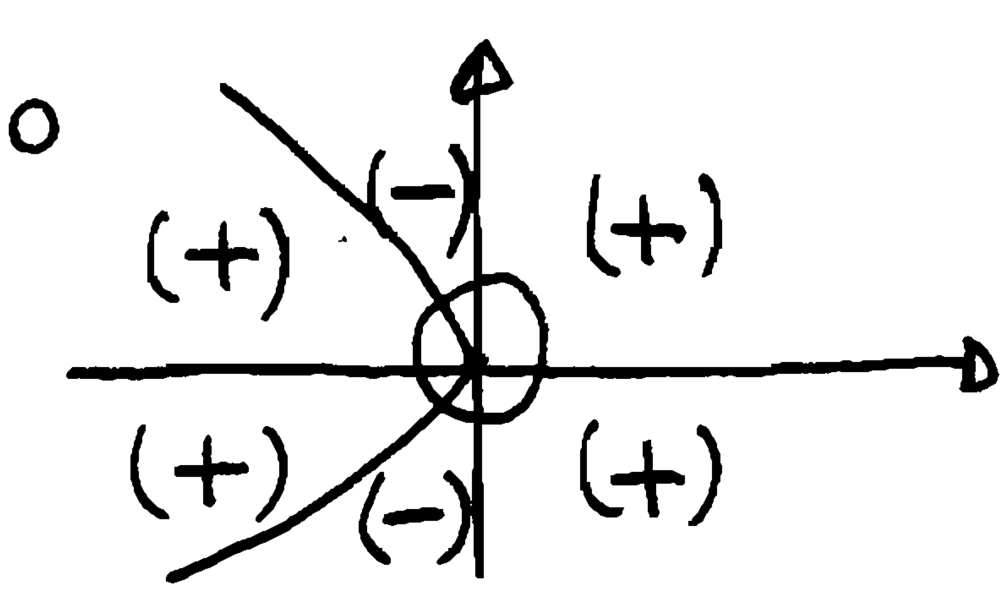
E insieme limitato e chiuso; $f(x;y)$ continua, vincolo qualificato.

$$\Lambda(x;y;\lambda) = x^2+xy^2-\lambda(4x^2+y^2-4)$$

Caso $\lambda = 0$ $\begin{cases} \Lambda'_x = 2x+y^2 = 0 \\ \Lambda'_y = 2xy = 0 \\ 4x^2+y^2 \leq 4 \end{cases} \Rightarrow \begin{cases} 2x = -y^2 \\ -y^3 = 0 \\ 4x^2+y^2 \leq 4 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \leq 4 \end{cases}$; $H(x;y) = \begin{Vmatrix} 2 & 2y \\ 2y & 2x \end{Vmatrix}$; $H(0;0) = \begin{Vmatrix} 2 & 0 \\ 0 & 0 \end{Vmatrix}$.

Sicuramente non è punto di Max. $f(0;0) = 0$; $f(x;y) = x(x+y^2) \geq 0$ per $\begin{cases} x \geq 0 \\ x \geq -y^2 \end{cases}$

In ogni intorno di $(0;0)$ risulta sia $f>0$ che $f<0 \Rightarrow (0;0)$ punto di Sella.



Caso $\lambda \neq 0$ $\begin{cases} \Lambda'_x = 2x+y^2-8\lambda x = 0 \\ \Lambda'_y = 2xy-2\lambda y = 2y(x-\lambda) = 0 \\ 4x^2+y^2 = 4 \end{cases} \Rightarrow \begin{cases} y = 0 \\ \lambda = \frac{2x}{8x} = \frac{1}{4} \\ 4x^2 = 4 \Rightarrow x^2 = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda = \frac{1}{4} \end{cases}$ (Max?) $\cup \begin{cases} x = -1 \\ y = 0 \\ \lambda = \frac{1}{4} \end{cases}$ (Max?)

$\cup \begin{cases} x = \lambda \\ 2x+y^2-8x^2 = 0 \\ 4x^2+y^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda = x \\ y^2 = 4-4x^2 \\ 2x+4-4x^2-8x^2 = 0 \end{cases} \Rightarrow 12x^2-2x-4 = 2(6x^2-x-2) = 0 \Rightarrow x = \frac{1 \pm \sqrt{1+48}}{12} \Rightarrow$

$\Rightarrow x = \frac{1\pm 7}{12} = \begin{cases} \frac{2}{3} \\ -\frac{1}{2} \end{cases} \Rightarrow$

AM3

$$\Rightarrow \begin{cases} x=\frac{2}{3} \\ y^2=4-\frac{16}{9}=\frac{20}{9} \\ \lambda=\frac{2}{3} \end{cases} \Rightarrow \underset{(\text{Max ?})}{\begin{cases} x=\frac{2}{3} \\ y=\frac{2\sqrt{5}}{3} \\ \lambda=\frac{2}{3} \end{cases}} \cup \underset{(\text{Max ?})}{\begin{cases} x=\frac{2}{3} \\ y=-\frac{2\sqrt{5}}{3} \\ \lambda=\frac{2}{3} \end{cases}} \cup \begin{cases} x=-\frac{1}{2} \\ y^2=4-1=3 \\ \lambda=-\frac{1}{2} \end{cases} \Rightarrow \underset{(\text{Min ?})}{\begin{cases} x=-\frac{1}{2} \\ y=\sqrt{3} \\ \lambda=-\frac{1}{2} \end{cases}} \cup \underset{(\text{Min ?})}{\begin{cases} x=-\frac{1}{2} \\ y=-\sqrt{3} \\ \lambda=-\frac{1}{2} \end{cases}}.$$

Studiamo la funzione sul bordo ponendo $(x=\cos t\,;\ y=2\,\mathrm{sen}\,t) \Rightarrow$

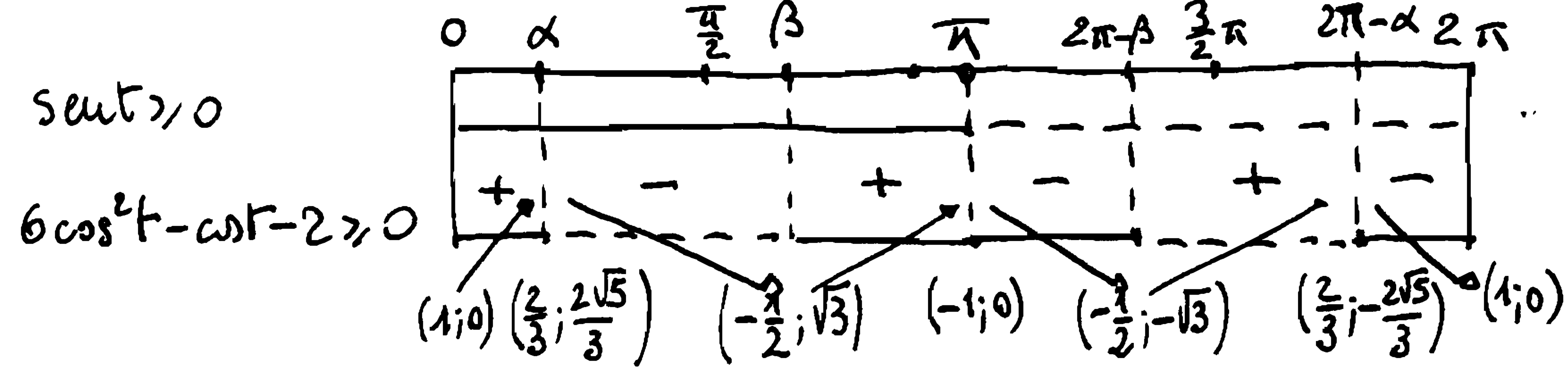
$\Rightarrow f(t)=\cos^2 t+4\cdot\cos t\cdot \mathrm{sen}^2 t \Rightarrow f'(t)=2\cos t\cdot(-\mathrm{sen}\,t)+4(-\mathrm{sen}^3 t+2\,\mathrm{sen}\,t\cos^2 t)=$

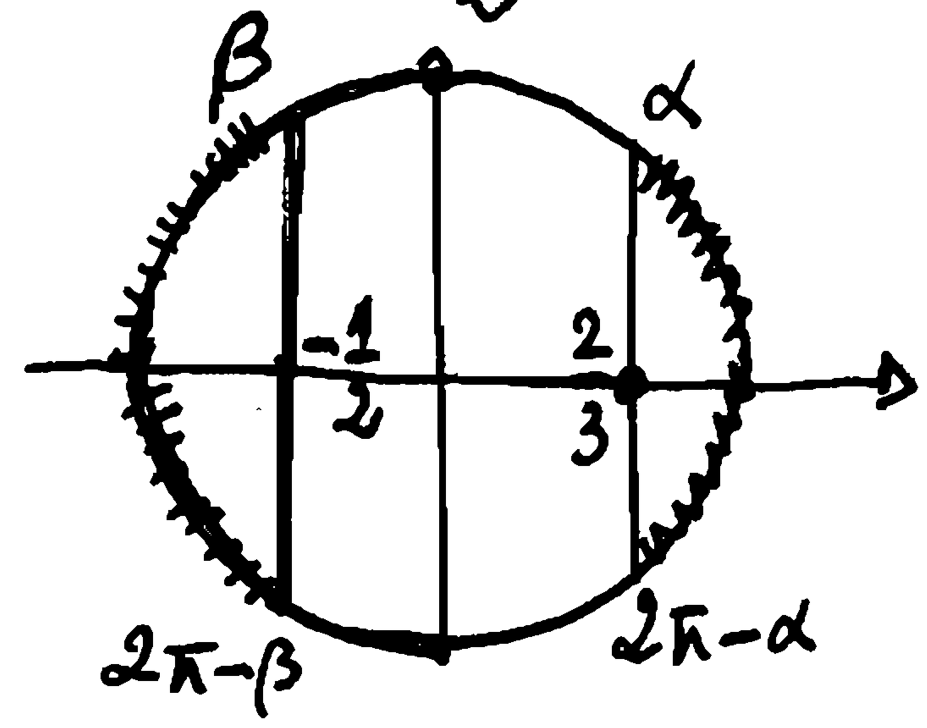
$=-2\,\mathrm{sen}\,t\cos t-4\,\mathrm{sen}^3 t+8\,\mathrm{sen}\,t\cos^2 t=2\,\mathrm{sen}\,t\cdot(4\cos^2 t-\cos t-2\,\mathrm{sen}^2 t)=$

$=2\,\mathrm{sen}\,t\,(4\cos^2 t-\cos t-2(1-\cos^2 t))=2\,\mathrm{sen}\,t\cdot(6\cos^2 t-\cos t-2)\geq 0$

$\mathrm{sen}\,t\geq 0$ per $0\leq t\leq \pi$; $6\cos^2 t-\cos t-2\geq 0$: $\cos t=\frac{1\pm\sqrt{1+48}}{12}=\frac{1\pm 7}{12} \left\langle \begin{array}{l} \frac{2}{3} \\ -\frac{1}{2} \end{array}\right.$

$6\cos^2 t-\cos t-2\geq 0$ per $\cos t\leq -\frac{1}{2} \cup \cos t\geq \frac{2}{3}$.





Solo il punto $(1;0)$ non conferma quanto trovato con il moltiplicatore.

$P_1=\left(\frac{2}{3};\frac{2\sqrt{5}}{3}\right)$; $P_2=(-1;0)$; $P_3=\left(\frac{2}{3};-\frac{2\sqrt{5}}{3}\right)$ sono punti di Max con $f\left(\frac{2}{3};\pm\frac{2\sqrt{5}}{3}\right)=\frac{52}{27}$ e $f(-1;0)=1$ per cui P_1 e P_3 sono punti di Max Assoluto e P_2 di Max relativo.

$P_4=\left(-\frac{1}{2};\sqrt{3}\right)$ e $P_5=\left(-\frac{1}{2};-\sqrt{3}\right)$ con $f(P_4)=f(P_5)=-\frac{5}{4}$ sono punti di Min. assoluto.

II M2) $\begin{cases} y''-4y=3(y'+x) \\ y(0)=\frac{9}{16} \\ y'(0)=\frac{1}{4} \end{cases} \Rightarrow y''-3y'-4y=3x \Rightarrow \lambda^2-3\lambda-4=0 \Rightarrow \lambda=\frac{3\pm\sqrt{9+16}}{2}=\frac{3\pm 5}{2}=\left\langle \begin{array}{l} 4 \\ -1 \end{array}\right.$.

Soluzione Eq. Omogenea: $y(x)=c_1e^{4x}+c_2e^{-x}$.

Poniamo $y_0(x)=ax+b \Rightarrow y_0'=a$; $y_0''=0 \Rightarrow 0-3a-4ax-4b=3x \Rightarrow \begin{cases} -4a=3 \\ -3a-4b=0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} a=-\frac{3}{4} \\ b=-\frac{3}{4}a=\frac{9}{16} \end{cases}$. Soluzione Generale della non omogenea: $y(x)=c_1e^{4x}+c_2e^{-x}-\frac{3}{4}x+\frac{9}{16}$.

$\boxed{AM4}$

$$\Rightarrow y'(x) = 4c_1 e^{4x} - c_2 e^{-x} - \frac{3}{4} \Rightarrow \begin{cases} y(0) = \frac{9}{16} \\ y'(0) = \frac{1}{4} \end{cases} \Rightarrow \begin{cases} c_1 + c_2 = 0 \\ 4c_1 - c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{1}{5} \\ c_2 = -\frac{1}{5} \end{cases}.$$

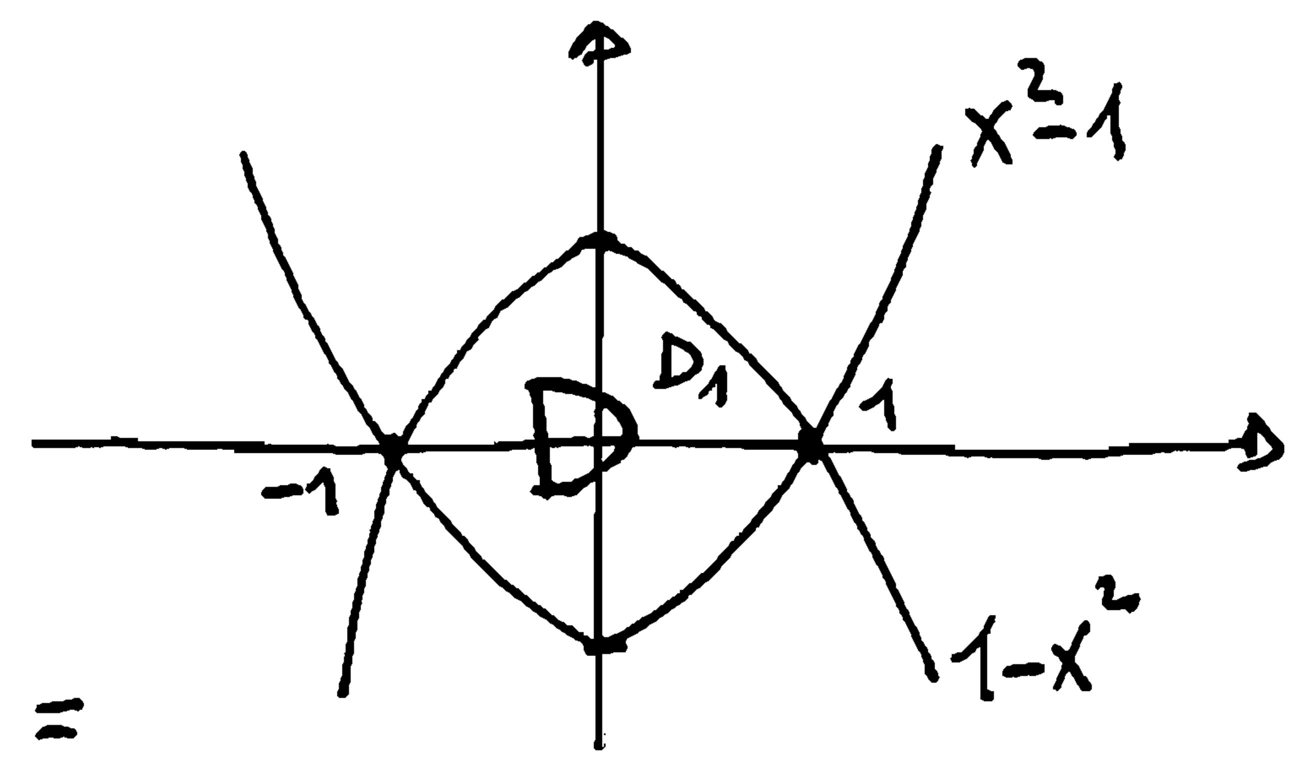
Soluzione del problema di Cauchy: $y(x) = \frac{1}{5} e^{4x} - \frac{1}{5} e^{-x} - \frac{3}{4} x + \frac{9}{16}$.

II M3) $\begin{cases} x' = y \\ y' = x \end{cases} \Rightarrow \begin{cases} x' - y = 0 \\ -x + y' = 0 \end{cases} \Rightarrow \begin{vmatrix} D & -1 \\ -1 & D \end{vmatrix}(x) = 0 \Rightarrow (D^2 - 1)(x) = 0 \Rightarrow x'' - x = 0.$

Soluzione: $x(t) = c_1 e^{t} + c_2 e^{-t}$. Ma $y(t) = x' = c_1 e^{t} - c_2 e^{-t}$.

Soluzione Generale: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix} + c_2 \begin{pmatrix} e^{t} \\ -e^{-t} \end{pmatrix}$.

II M4) $\iint_D x^2 + y^2 \, dx\, dy$ con $D = \{(x;y) : x^2 - 1 \leq y \leq 1 - x^2\}$



$$\iint_D x^2 + y^2 \, dx\,dy = \int_{-1}^{+1} \int_{x^2-1}^{1-x^2} (x^2 + y^2)\, dy\, dx = \int_{-1}^{+1} \left(x^2 y + \frac{1}{3} y^3 \right|_{x^2-1}^{1-x^2} dx =$$

$$= \int_{-1}^{+1} x^2(1-x^2) + \frac{1}{3}(1-x^2)^3 - x^2(x^2-1) - \frac{1}{3}(x^2-1)^3 \, dx = \int_{-1}^{+1} 2x^2(1-x^2) + \frac{2}{3}(1-x^2)^3 \, dx =$$

$$= \int_{-1}^{+1} 2x^2 - 2x^4 + \frac{2}{3}(1 - 3x^2 + 3x^4 - x^6)\, dx = \left(\cancel{\frac{2}{3}x^3} - \cancel{\frac{2}{5}x^5} + \frac{2}{3}x - \cancel{\frac{2}{3}x^3} + \cancel{\frac{2}{5}x^5} - \frac{2}{21}x^7 \right|_{-1}^{1} =$$

$$= \left(\frac{2}{3}x - \frac{2}{21}x^7 \right|_{-1}^{1} = \left(\frac{2}{3} - \frac{2}{21} \right) - \left(-\frac{2}{3} + \frac{2}{21} \right) = \frac{4}{3} - \frac{4}{21} = \frac{24}{21} = \frac{8}{7}.$$

Date le evidenti simmetrie, si sarebbe potuto calcolare:

$$\iint_D x^2 + y^2 \, dx\,dy = 4 \iint_{D_1} x^2 + y^2 \, dx\,dy = 4 \int_0^1 \int_0^{1-x^2} x^2 + y^2 \, dy\, dx = \frac{8}{7}.$$