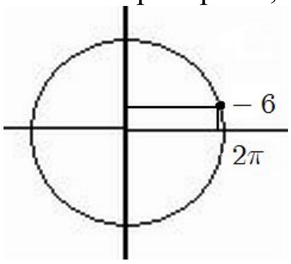
TASK MATHEMATICS for ECONOMIC APPLICATIONS 16/03/2017

I M 1) Using the definition of a complex exponential we get:
$z = e^{1-6i} = e \cdot e^{-6i} = e(\cos(-6) + i\sin(-6)) = e\,(\cos 6 - i\sin 6) =$
$= e\cos 6 - i\,e\sin 6$, the algebraic form of z.
Since $2\pi \cong 6,28$ the number z, in the complex plane, has this position:



I M 2) To achieve the requested matrix firstly we calculate its characteristic polynomial:

$$p_{\mathbb{A}}(\lambda) = |\,\mathbb{A} - \lambda\mathbb{I}\,| = \begin{vmatrix} 2-\lambda & 2 & 3 \\ 1 & 3-\lambda & 3 \\ 1 & 2 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & \lambda-1 \\ 1 & 3-\lambda & 3 \\ 1 & 2 & 4-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\begin{vmatrix} 3-\lambda & 3 \\ 2 & 4-\lambda \end{vmatrix} + (\lambda-1)\begin{vmatrix} 1 & 3-\lambda \\ 1 & 2 \end{vmatrix} =$$

$$= (1-\lambda)\big(\lambda^2 - 7\lambda + 6 + (1-\lambda)\big) = (1-\lambda)(\lambda-1)(\lambda-7)\,.$$

The three eigenvalues of $\mathbb{A}$ are $\lambda_1 = \lambda_2 = 1\,,\ \lambda_3 = 7\,.$

Since $||\,\mathbb{A} - 1\cdot\mathbb{I}\,|| = \begin{Vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{Vmatrix}$ it follows $m_1^g = 3 - \text{Rank}\,(\mathbb{A} - 1\cdot\mathbb{I}) = 2$ and so the matrix is a diagonalizable one.

The second step is to find for every eigenvalue the corresponding eigenvectors.
For $\lambda_{1,2} = 1$ we solve the system:

$$(\mathbb{A} - \lambda_1\mathbb{I})\cdot\mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ x_1 + 2x_2 + 3x_3 = 0 \Rightarrow \\ x_1 + 2x_2 + 3x_3 = 0 \end{cases}$$

$\Rightarrow x_1 = -2x_2 - 3x_3 \Rightarrow \mathbb{X} = (-2x_2 - 3x_3, x_2, x_3)$. Two linearly independent eigenvectors corresponding to $\lambda = 1$ are $\mathbb{X}_1 = (-2, 1, 0)$ and $\mathbb{X}_2 = (-3, 0, 1)$.
For $\lambda_3 = 7$ we solve the system:

$$(\mathbb{A} - \lambda_3\mathbb{I})\cdot\mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} -5 & 2 & 3 \\ 1 & -4 & 3 \\ 1 & 2 & -3 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} -5x_1 + 2x_2 + 3x_3 = 0 \\ x_1 - 4x_2 + 3x_3 = 0 \\ x_1 + 2x_2 - 3x_3 = 0 \end{cases} \Rightarrow \begin{cases} 12x_2 - 12x_3 = 0 \\ -6x_2 + 6x_3 = 0 \\ x_1 = -2x_2 + 3x_3 \end{cases} \Rightarrow \begin{cases} x_1 = x_3 \\ x_2 = x_3 \end{cases}$$

and so the eigenvector corresponding to $\lambda = 7$ is $\mathbb{X}_3 = (x, x, x) \Rightarrow \mathbb{X}_3 = (1, 1, 1)\,.$

So a matrix satisfying $\mathbb{A}\cdot\mathbb{P} = \mathbb{P}\cdot\mathbb{D}$ is $\mathbb{P} = \begin{Vmatrix} -2 & -3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{Vmatrix}$.

I M 3) *First method, by the transpose of the adjoint matrix divided by the determinant.*

$$|\mathbb{A}| = \begin{vmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 3 \\ 0 & 1 & -1 \\ 1 & 2 & 4 \end{vmatrix} = 2\cdot\begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} + 1\cdot\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 12 - 5 = 7\,.$$

(note that $|\mathbb{A}| = 1\cdot 1\cdot 7$, the product of the eingenvalues of the matrix). And so:

$$Adj(\mathbb{A}) = \left\| \begin{matrix} \begin{vmatrix} 3 & 3 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 3 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} \end{matrix} \right\| = \left\| \begin{matrix} 6 & -1 & -1 \\ -2 & 5 & -2 \\ -3 & -3 & 4 \end{matrix} \right\|.$$

$$\mathbb{A}^{-1} = \frac{1}{|\mathbb{A}|}(Adj(\mathbb{A}))^T = \frac{1}{7}\left\| \begin{matrix} 6 & -2 & -3 \\ -1 & 5 & -3 \\ -1 & -2 & 4 \end{matrix} \right\| = \left\| \begin{matrix} 6/7 & -2/7 & -3/7 \\ -1/7 & 5/7 & -3/7 \\ -1/7 & -2/7 & 4/7 \end{matrix} \right\|.$$

Second method, by elementary operations on $the\, lines\, of\, the$ matrix.

$$\left\| \begin{matrix} 2 & 2 & 3 & | & 1 & 0 & 0 \\ 1 & 3 & 3 & | & 0 & 1 & 0 \\ 1 & 2 & 4 & | & 0 & 0 & 1 \end{matrix} \right\| \overset{\substack{R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{1}{2}R_1}}{\equiv} \left\| \begin{matrix} 2 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 2 & 3/2 & | & -1/2 & 1 & 0 \\ 0 & 1 & 5/2 & | & -1/2 & 0 & 1 \end{matrix} \right\| \overset{R_3 - \frac{1}{2}R_2}{\equiv}$$

$$\left\| \begin{matrix} 2 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 2 & 3/2 & | & -1/2 & 1 & 0 \\ 0 & 0 & 7/4 & | & -1/4 & -1/2 & 1 \end{matrix} \right\| \overset{\substack{\frac{1}{2}R_1 \\ \frac{1}{2}R_2 \\ \frac{4}{7}R_3}}{\equiv}$$

$$\left\| \begin{matrix} 1 & 1 & 3/2 & | & 1/2 & 0 & 0 \\ 0 & 1 & 3/4 & | & -1/4 & 1/2 & 0 \\ 0 & 0 & 1 & | & -1/7 & -2/7 & 4/7 \end{matrix} \right\| \overset{\substack{R_1 - R_2 \\ R_2 - \frac{3}{4}R_3}}{\equiv}$$

$$\left\| \begin{matrix} 1 & 0 & 3/4 & | & 3/4 & -1/2 & 0 \\ 0 & 1 & 0 & | & -1/7 & 5/7 & -3/7 \\ 0 & 0 & 1 & | & -1/7 & -2/7 & 4/7 \end{matrix} \right\| \overset{R_1 - \frac{3}{4}R_3}{\equiv}$$

$$\left\| \begin{matrix} 1 & 0 & 0 & | & 6/7 & -2/7 & -3/7 \\ 0 & 1 & 0 & | & -1/7 & 5/7 & -3/7 \\ 0 & 0 & 1 & | & -1/7 & -2/7 & 4/7 \end{matrix} \right\|.$$

In the right block we have the requested inverse matrix.

I M 4) If $\mathbb{A} = \left\| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right\|$, since $\mathbb{X}_1$ and $\mathbb{X}_2$ belong to the Kernel, we have:

$$\mathbb{A} \cdot \mathbb{X}_1 = \mathbb{O} \Rightarrow \left\| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right\| \left\| \begin{matrix} 0 \\ 1 \\ 1 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\| \Rightarrow \begin{cases} a_{12} + a_{13} = 0 \\ a_{22} + a_{23} = 0 \Rightarrow \\ a_{32} + a_{33} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_{13} = -a_{12} \\ a_{23} = -a_{22} \Rightarrow \mathbb{A} = \\ a_{33} = -a_{32} \end{cases} \left\| \begin{matrix} a_{11} & a_{12} & -a_{12} \\ a_{21} & a_{22} & -a_{22} \\ a_{31} & a_{32} & -a_{32} \end{matrix} \right\|;$$

$$\mathbb{A} \cdot \mathbb{X}_2 = \mathbb{O} \Rightarrow \left\| \begin{matrix} a_{11} & a_{12} & -a_{12} \\ a_{21} & a_{22} & -a_{22} \\ a_{31} & a_{32} & -a_{32} \end{matrix} \right\| \left\| \begin{matrix} 1 \\ 1 \\ 0 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right\| \Rightarrow \begin{cases} a_{11} + a_{12} = 0 \\ a_{21} + a_{22} = 0 \Rightarrow \\ a_{31} + a_{32} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a_{11} = -a_{12} \\ a_{21} = -a_{22}, \mathbb{A} = \\ a_{31} = -a_{32} \end{cases} \left\| \begin{matrix} -a_{12} & a_{12} & -a_{12} \\ -a_{22} & a_{22} & -a_{22} \\ -a_{32} & a_{32} & -a_{32} \end{matrix} \right\|.$$

Finally, from $f(1,1,1) = (2,1,-1)$, $\mathbb{A}$ must satisfy:

$$\left\|\begin{matrix} -a_{12} & a_{12} & -a_{12} \\ -a_{22} & a_{22} & -a_{22} \\ -a_{32} & a_{32} & -a_{32} \end{matrix}\right\| \left\|\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}\right\| = \left\|\begin{matrix} 2 \\ 1 \\ -1 \end{matrix}\right\| \Rightarrow \begin{cases} -a_{12} = 2 \\ -a_{22} = 1 \\ -a_{32} = -1 \end{cases} \Rightarrow \begin{cases} a_{12} = -2 \\ a_{22} = -1 \\ a_{32} = 1 \end{cases}.$$

So $\mathbb{A} = \left\|\begin{matrix} 2 & -2 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{matrix}\right\|$.

To find a basis for the Kernel and a basis for the Image, since $\text{Rank}(\mathbb{A}) = 1$ we have simply to note that $\mathbb{X}_1$ and $\mathbb{X}_2$ are linear independent, so $\mathcal{B}_{Ker(f)} = \{\mathbb{X}_1, \mathbb{X}_2\}$ and $\mathcal{B}_{Imm(f)} = \{(2, 1, -1)\}$.

II M 1) The equation $f(x,y,z) = xe^{y+z} - ye^{x+z} + ze^{x+y} = 0$ verifies $f(1,1,0) = 0$. $\nabla f = (e^{y+z} - ye^{x+z} + ze^{x+y}; xe^{y+z} - e^{x+z} + ze^{x+y}; xe^{y+z} - ye^{x+z} + e^{x+y})$ from which we get $\nabla f(1,1,0) = (0;0;e^2)$. Since $f'_z(P) \neq 0$, the equation defines an implicit function $z = z(x,y)$ with:

$$\nabla z(1,1) = \left(-\frac{f'_x(P)}{f'_z(P)}; -\frac{f'_y(P)}{f'_z(P)}\right) = \left(-\frac{0}{e^2}; -\frac{0}{e^2}\right) = (0,0).$$

II M 2) The problem $\begin{cases} \text{Max/min } f(x,y) = x^2 + xy^2 \\ \text{u.c. } 4x^2 + y^2 \leq 4 \end{cases}$ is equivalent to the problem:

$\begin{cases} \text{Max/min } f(x,y) = x^2 + xy^2 \\ \text{u.c. } 4x^2 + y^2 - 4 \leq 0 \end{cases}$; the feasible region is drawn in red in the figure on the next page; the objective function is continous and the feasible region is a bounded and closed set, so by Weiertrass Theorem the problem has absolute maximum and minimum. The Lagrangian function of the problem is:

$\Lambda(x,y,\lambda) = x^2 + xy^2 - \lambda\left(4x^2 + y^2 - 4\right)$

whose gradient vector is $\nabla\Lambda = \left(2x + y^2 - 8\lambda x; 2xy - 2\lambda y; -\left(4x^2 + y^2 - 4\right)\right)$.

KUNH-TUCKER CONDITIONS

First case (*free optimization*): $\begin{cases} \lambda = 0 \\ 2x + y^2 = 0 \\ 2xy = 0 \\ 4x^2 + y^2 - 4 \leq 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ x = 0 \\ y = 0 \\ 0 \leq 4 \end{cases}$, the unique solution point is $(0,0)$. $\mathbb{H}(f) = \left\|\begin{matrix} 2 & 2y \\ 2y & 2x \end{matrix}\right\|$, $\mathbb{H}(0,0) = \left\|\begin{matrix} 2 & 0 \\ 0 & 0 \end{matrix}\right\|$ with $|\mathbb{H}(0,0)| = 0$; second order conditions are useless to give us informations on the nature of the point $(0,0)$, but from the figure on the next page it is easy to see that $(0,0)$ is a saddle point for the objective function of the problem.

Second case (*active constraint*):

$\begin{cases} \lambda \neq 0 \\ 2x + y^2 - 8\lambda x = 0 \\ 2xy - 2\lambda y = 0 \\ 4x^2 + y^2 - 4 = 0 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ 2x + y^2 - 8\lambda x = 0 \\ 2y(x - \lambda) = 0 \\ 4x^2 + y^2 = 4 \end{cases}$ from which we get:

$$\begin{cases} \lambda \neq 0 \\ 2x(1-4\lambda)=0 \\ y=0 \\ 4x^2=4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ \lambda = 1/4 \\ y=0 \\ x=\pm 1 \end{cases} \text{ and}$$

$$\begin{cases} \lambda \neq 0 \\ y^2 = 8\lambda^2 - 2\lambda \\ x=\lambda \\ 12\lambda^2 - 2\lambda = 4 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ y^2 = 8\lambda^2 - 2\lambda \\ x=\lambda \\ (2\lambda+1)(3\lambda-2)=0 \end{cases} \text{ from which we get:}$$

$$\begin{cases} \lambda \neq 0 \\ y^2 = 3 \\ x = -1/2 \\ \lambda = -1/2 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ y = \pm\sqrt{3} \\ x = -1/2 \\ \lambda = -1/2 \end{cases} \text{ and } \begin{cases} \lambda \neq 0 \\ y^2 = 20/9 \\ x = 2/3 \\ \lambda = 2/3 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ y = \pm\sqrt{20}/3 \\ x = 2/3 \\ \lambda = 2/3 \end{cases}.$$

So we get six constrained critical points:

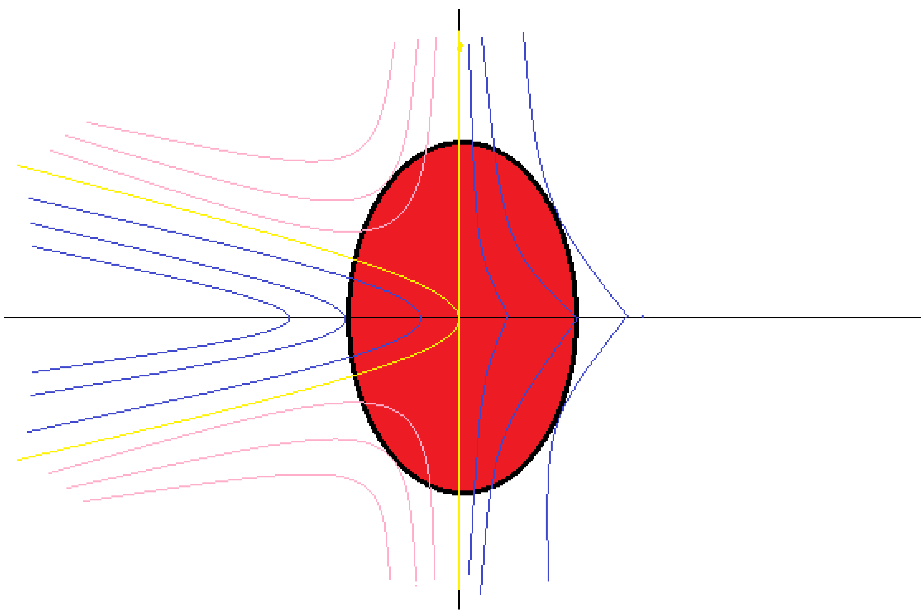
$P_{1,2} = (\pm 1, 0)$, $P_{3,4} = \left(2/3, \pm\sqrt{20}/3\right)$, $Max\,?$; $P_{5,6} = \left(-1/2, \pm\sqrt{3}\right)$, $min\,?$.

$f(P_{1,2}) = 1$, $f(P_{3,4}) = 44/9$, $f(P_{5,6}) = -5/4$.

$Max(f) = 44/9$ at points $P_{3,4}$; $minf = -5/4$ at points $P_{5,6}$.

Note also that by figure below point P_2 is a local maximum; P_1 is not a maximum point.

On the figure below there are drawn zero level curve (yellow), positive level curves (blue) and negative level curves (pink).



II M 3) For $f(x,y,z) = \log(x-y) - e^{z-x} + x^2y - yz^3$, differentiable function, we have: $D_v f(P_0) = \nabla f(P_0) \cdot v$, where v is the unit vector of $(1,1,1)$. $P_0 = (1,0,1)$

$$\nabla f(x,y,z) = \left(\frac{1}{x-y} + e^{z-x} + 2xy\,;\, \frac{-1}{x-y} + x^2 - z^3\,;\, -e^{z-x} - 3yz^2\right)$$

$\nabla f(1,0,1) = (1+1+0\,;\,-1+1-1\,;\,-1-0) = (2;\,-1;\,-1)$. Since

$v = \left(\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}}\right)$ we get $D_v f(P_0) = (2;\,-1;\,-1)\cdot\left(\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}}\right) = 0$.

II M 4) The Mac Laurin polynomial of degree 2 at point $(0,0)$ for the function f is

$P_2(x,y) = f(0,0) + \nabla f(0,0)\cdot(x,y) + \frac{1}{2}(x,y)\cdot \mathbb{H}f(0,0)\cdot \left\|\begin{matrix} x \\ y \end{matrix}\right\|$; $f(0,0) = 0$,

$$\nabla f = \left(e^{(x-y)} + (x+y)\,e^{(x-y)}; e^{(x-y)} - (x+y)\,e^{(x-y)}\right) =$$

$$\nabla f = \left((1+x+y)e^{(x-y)}; (1-x-y)e^{(x-y)}\right);\ \nabla f(0,0) = (1,1);$$

$$\mathbb{H}f(x,y) = \left\|\begin{matrix} e^{(x-y)} + (1+x+y)e^{(x-y)} & e^{(x-y)} - (1+x+y)e^{(x-y)} \\ -e^{(x-y)} + (1-x-y)e^{(x-y)} & -e^{(x-y)} - (1-x-y)e^{(x-y)} \end{matrix}\right\| =$$

$$\mathbb{H}f(x,y) = \left\|\begin{matrix} (2+x+y)e^{(x-y)} & -(x+y)e^{(x-y)} \\ -(x+y)e^{(x-y)} & -(2-x-y)e^{(x-y)} \end{matrix}\right\|,\ \mathbb{H}f(0,0) = \left\|\begin{matrix} 2 & 0 \\ 0 & -2 \end{matrix}\right\|.$$

So $P_2(x,y) = x + y + \frac{1}{2}\left(2x^2 - 2y^2\right) = x + y + x^2 - y^2$.