TASK MATHEMATICS for ECONOMIC APPLICATIONS
12/06/2017

I M 1) $z^4 - 3i\,z^3 - z^2 - 3i\,z - 2 = z^4 - 3i\,z^3 - 2z^2 + z^2 - 3i\,z - 2 =$
$= z^2\big(z^2 - 3i\,z - 2\big) + \big(z^2 - 3i\,z - 2\big) = \big(z^2 - 3i\,z - 2\big)\cdot\big(z^2+1\big) =$
$= (z-2i)\cdot(z-i)\cdot(z-i)\cdot(z+i) = (z-2i)\cdot(z-i)^2\cdot(z+i)$;
so $z^4 - 3i\,z^3 - z^2 - 3i\,z - 2 = 0$ if and only if $(z-2i)\cdot(z-i)^2\cdot(z+i) = 0$ and the four roots are $z_1 = 2i$, $z_2 = z_3 = i$, $z_4 = -i$ with modulus $|z_1| = 2$, $|z_2| = |z_3| = |z_4| = 1$. z_1 is the solution having the maximum modulus and its square roots are $\sqrt{z_1} = \sqrt{2i} = \sqrt{2(\cos(\pi/2) + i\sin(\pi/2))} =$
$= \sqrt{2}\,(\cos(\pi/4 + k\pi) + i\sin(\pi/4 + k\pi))$; $k = 0, 1$;
so the two roots are:
$w_1 = \sqrt{2}\,(\cos(\pi/4) + i\sin(\pi/4)) = 1 + i$ and
$w_2 = \sqrt{2}\,(\cos(5\pi/4) + i\sin(5\pi/4)) = -1 - i = -w_1$.

I M 2) The characteristic polynomial of $\mathbb{A}$ is $p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| =$

$$= \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & k & 0 \\ 0 & k & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & k & 0 \\ k & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} - \begin{vmatrix} 1 & k & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} =$$

$$= \lambda^2 \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} + k\lambda \begin{vmatrix} k & 1 \\ 0 & -\lambda \end{vmatrix} - \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} =$$

$$= \lambda^2\big(\lambda^2 - 1\big) - k^2\lambda^2 - \big(\lambda^2 - 1\big) = \big(\lambda^2 - 1\big)^2 - k^2\lambda^2 ;$$

since $\lambda = 1$ is an eigevalue of $\mathbb{A}$, $p_{\mathbb{A}}(1) = -k^2 = 0$ and this implies $k = 0$; the characteristic polyinomial of $\mathbb{A}$ becomes $p_{\mathbb{A}}(\lambda) = \big(\lambda^2 - 1\big)^2 = (\lambda - 1)^2\cdot(\lambda + 1)^2$ with eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = -1$. To find an orthogonal matrix which diagonalizes $\mathbb{A}$ we must find a matrix $\mathbb{P} = \| \ \mathcal{V}_1 \quad \mathcal{V}_2 \quad \mathcal{V}_3 \quad \mathcal{V}_4 \ \|$ where $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$ are four eigenvectors two by two orthogonal with $\mathcal{V}_1$ and $\mathcal{V}_2$ associated to the eigenvalue $\lambda = 1$ and $\mathcal{V}_3$ and $\mathcal{V}_4$ associated to eigenvalue $\lambda = -1$.

For $\mathcal{V}_1$ and $\mathcal{V}_2$ we consider a generic element $\mathcal{V} = \left\| \begin{matrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{matrix} \right\|$ which must satisfy the condition $(\mathbb{A} - \mathbb{I})\cdot\mathcal{V} = \mathbb{O}$ or, in system form, $\begin{cases} -v_1 + v_2 = 0 \\ v_1 - v_2 = 0 \\ -v_3 + v_4 = 0 \\ v_3 - v_4 = 0 \end{cases} \Rightarrow \begin{cases} v_1 = v_2 \\ v_3 = v_4 \end{cases}$; so we get

$\mathcal{V}_1 = \left\| \begin{matrix} 1 \\ 1 \\ 0 \\ 0 \end{matrix} \right\|$, $\mathcal{V}_2 = \left\| \begin{matrix} 0 \\ 0 \\ 1 \\ 1 \end{matrix} \right\|$. Similarly, for $\mathcal{V}_3$ and $\mathcal{V}_4$ a generic $\mathcal{V}$ must satisfy the condition $(\mathbb{A} + \mathbb{I})\cdot\mathcal{V} = \mathbb{O}$ or, in system form, $\begin{cases} v_1 + v_2 = 0 \\ v_1 + v_2 = 0 \\ v_3 + v_4 = 0 \\ v_3 + v_4 = 0 \end{cases} \Rightarrow \begin{cases} v_1 = -v_2 \\ v_3 = -v_4 \end{cases}$; so we get

$\mathcal{V}_3 = \begin{Vmatrix} 1 \\ -1 \\ 0 \\ 0 \end{Vmatrix}$, $\mathcal{V}_4 = \begin{Vmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Vmatrix}$, and finally the orthogonal matrix, having the unit vectors as columns, is $\mathbb{P} = \begin{Vmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{Vmatrix}$.

I M 3) If vector $\mathbb{Y}$ has coordinates $(2, -1, 1)$ in the basis $\mathbb{W}$, it is:

$\mathbb{Y} = 2\begin{Vmatrix} 1 \\ 2 \\ 1 \end{Vmatrix} - \begin{Vmatrix} 2 \\ -1 \\ 0 \end{Vmatrix} + \begin{Vmatrix} k \\ k \\ k \end{Vmatrix} = \begin{Vmatrix} k \\ 5+k \\ 2+k \end{Vmatrix} \Leftrightarrow \begin{Vmatrix} -1 \\ 4 \\ 1 \end{Vmatrix} = \begin{Vmatrix} k \\ 5+k \\ 2+k \end{Vmatrix}$, so $k = -1$.

I M 4) By a reduction of the system with elementary operations on the rows we get:

$$\begin{Vmatrix} 1 & 2 & 0 & -1 & | & 2 \\ 2 & 1 & -1 & m & | & k \\ -1 & 4 & 2k & m & | & 4 \end{Vmatrix} \overset{\substack{R_2-2R_1 \\ R_3+R_1}}{\Rightarrow} \begin{Vmatrix} 1 & 2 & 0 & -1 & | & 2 \\ 0 & -3 & -1 & m+2 & | & k-4 \\ 0 & 6 & 2k & m-1 & | & 6 \end{Vmatrix} \overset{R_3+2R_2}{\Rightarrow}$$

$\Rightarrow \begin{Vmatrix} 1 & 2 & 0 & -1 & | & 2 \\ 0 & -3 & -1 & m+2 & | & k-4 \\ 0 & 0 & 2(k-1) & 3(m+1) & | & 2(k-1) \end{Vmatrix}$. For the last matrix we can observe that when $k \neq 1$ or $m \neq -1$, both matrix and augmented matrix have the same rank equal to 3 and when $k = 1$ and $m = -1$ both matrix and augmented matrix have the same rank equal to 2; we conclude that for any pair k, m the system admits solutions and the number of solutions is ∞^2 if $k = 1$ and $m = -1$, ∞^1 otherwise.

II M 1) $f(P) = 3$ and $g(P) = 1$, conditions are satisfied at point P. Firstly we calculate the Jacobian matrix of f and g :

$\frac{\partial(f,g)}{\partial(x,y,z)} = \mathbb{J} = \begin{Vmatrix} 2x & 2y & 2z \\ y+z & x-z & -y+x \end{Vmatrix}$ with $\mathbb{J}(P) = \begin{Vmatrix} -2 & 2 & -2 \\ 0 & 0 & -2 \end{Vmatrix}$.

Since $\left|\mathbb{J}(P)_{(x,y)}\right| = \begin{vmatrix} -2 & 2 \\ 0 & 0 \end{vmatrix} = 0$, $\left|\mathbb{J}(P)_{(x,z)}\right| = \begin{vmatrix} -2 & -2 \\ 0 & -2 \end{vmatrix} = 4$ and $\left|\mathbb{J}(P)_{(y,z)}\right| = \begin{vmatrix} 2 & -2 \\ 0 & -2 \end{vmatrix} = -4$, the system $\begin{cases} f(x,y,z) = x^2+y^2+z^2 = 3 \\ g(x,y,z) = xy - yz + xz = 1 \end{cases}$ at point $P = (-1, 1, -1)$ can define an implicit function of the type $(y,z) \mapsto x(y,z)$ or $(x,z) \mapsto y(x,z)$. If we consider the first type of function we get:

$\frac{\partial x}{\partial y} = -\frac{\begin{vmatrix} -2 & -2 \\ 0 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & -2 \\ 0 & -2 \end{vmatrix}} = 1$ and $\frac{\partial x}{\partial z} = -\frac{\begin{vmatrix} 2 & -2 \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & -2 \\ 0 & -2 \end{vmatrix}} = 0$, while for the second type

$\frac{\partial y}{\partial x} = -\frac{\begin{vmatrix} 2 & -2 \\ 0 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & -2 \\ 0 & -2 \end{vmatrix}} = 1$ and $\frac{\partial y}{\partial z} = -\frac{\begin{vmatrix} -2 & 2 \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} -2 & -2 \\ 0 & -2 \end{vmatrix}} = 0$.

II M 2) For the Problem $\begin{cases} \text{Max/min } f(x,y) = x^2 + y^3 \\ \text{u.c.} \begin{cases} x \geq 0 \\ y \geq 0 \\ y \leq 1 - x \end{cases} \end{cases}$ the admissible region is red drawed in the figure in the next page; note that the objective function is continous and the admissible region is a bounded and closed set, so by Weierstrass Theorem the problem admits absolute maximum and minimum value.

First case (*free optimization*): $\nabla f = \left(2x, 3y^2\right)$, the system $\begin{cases} 2x = 0 \\ 3y^2 = 0 \\ x \geq 0 \\ y \geq 0 \\ y \leq 1 - x \end{cases}$ has the unique solution $(0,0)$. $\mathbb{H}(f) = \left\|\begin{matrix} 2 & 0 \\ 0 & 6y \end{matrix}\right\|$, $\mathbb{H}(f(0,0)) = \left\|\begin{matrix} 2 & 0 \\ 0 & 0 \end{matrix}\right\|$ and $|\mathcal{H}f(0,0)| = 0$, so we cannot get any conclusion about point $(0,0)$.

Second case (*along the border of the admissible region*):

(a) if $x = 0$ we consider the function $g(y) = f(0,y) = y^3$, for $0 \leq y \leq 1$ $g(y)$ has minimum for $y = 0$ and maximum for $y = 1$;

(b) if $y = 0$ we consider the function $h(x) = f(x,0) = x^2$, for $0 \leq x \leq 1$ $h(x)$ has minimum for $x = 0$ and maximum for $x = 1$;

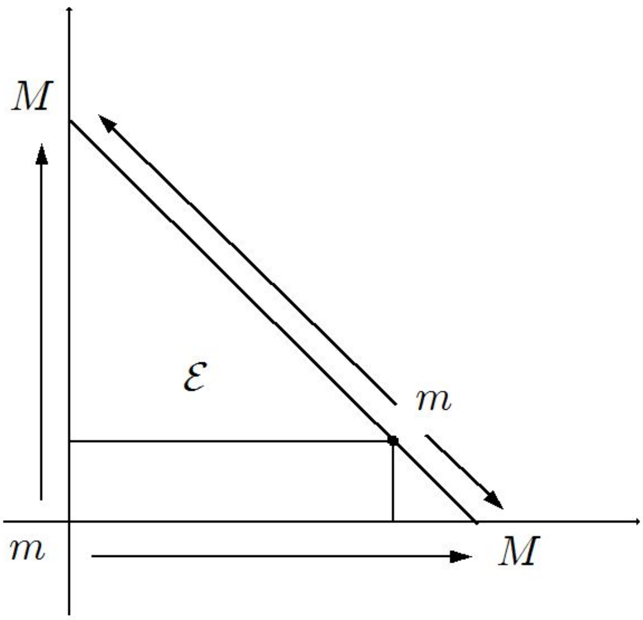
(c) finally if $y = 1 - x$, we consider the function :

$k(x) = f(x, 1-x) = x^2 + (1-x)^3 = -x^3 + 4x^2 - 3x + 1$,

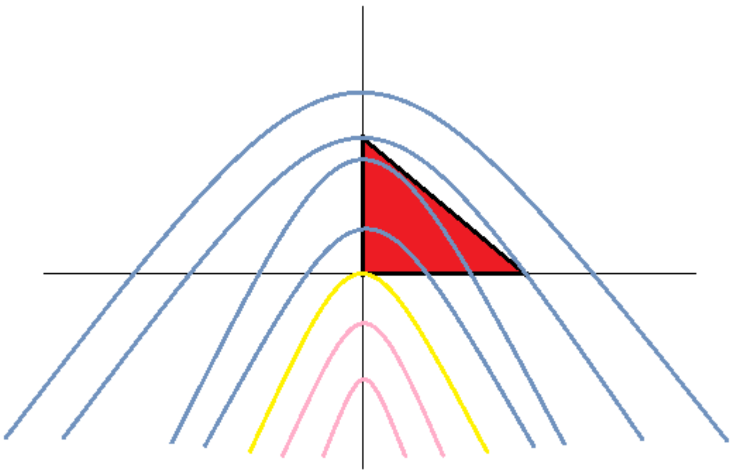
, $k'(x) = -3x^2 + 8x - 3$ with $k'(x) \geq 0$ for $\frac{2}{3}\left(4 - \sqrt{7}\right) \leq x \leq \frac{2}{3}\left(4 + \sqrt{7}\right)$.

For $0 \leq x \leq 1$ $k(x)$ has minimum for $x = \frac{2}{3}\left(4 - \sqrt{7}\right)$ and maximum for $x = 0$ or $x = 1$.

We conclude: $MAX(f) = MAX\{f(0,1), f(1,0)\} = 1$ at points $(0,1)$ and $(1,0)$ and $min(f) = min\left\{f(0,0), f\left(\frac{2}{3}\left(4 - \sqrt{7}\right), \frac{1}{3}\left(\sqrt{7} - 1\right)\right)\right\} = 0$ at point $(0,0)$.



On the figure below we draw zero level curve (yellow), positive level curves (blue) and negative level curves (pink).



II M 3) Since $f(x,y)$ is a differentiable function:
$D_v f(P_0) = \nabla f(P_0)\cdot v$ and $D^2_{v,v} f(P_0) = v\cdot \mathbb{H}(P_0)\cdot v^{\mathrm{T}}$. Now
$\nabla f = \left(2xy - y^2, x^2 - 2xy\right)$, $\nabla f(P_0) = (1,\,-1)$, and so
$D_v f(P_0) = (1,\,-1)\cdot(\cos\alpha, \sin\alpha) = \cos\alpha - \sin\alpha$.
Since $D_v f(P_0) = 0$ it follows $\cos\alpha - \sin\alpha = 0$. Then

$$\mathbb{H}(x,y) = \begin{Vmatrix} 2y & 2x-2y \\ 2x-2y & -2x \end{Vmatrix},\ \mathbb{H}(P_0) = \begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix}.$$

$D^2_{v,v} f(P_0) = 2\cos^2\alpha - 2\sin^2\alpha = 2(\cos\alpha - \sin\alpha)(\cos\alpha + \sin\alpha) = 0$
since $\cos\alpha - \sin\alpha = 0$.

II M 4) For $f(x,y,z) = \log\left(x^2 - y\right) + ze^{zx} + \cos(y - z)$ we get:

$$\nabla f = \left(\frac{2x}{x^2 - y} + z^2 e^{zx}, \frac{-1}{x^2 - y} - \sin(y-z), e^{zx} + xze^{zx} + \sin(y-z)\right),$$

and so $\nabla f(1,0,1) = (2 + e, \sin 1 - 1, 2e - \sin 1)$.