

TASK MATHEMATICS for ECONOMIC APPLICATIONS
11/07/2017

I M 1) $z^3 - z^2 + 4z - 4 = z^2(z - 1) + 4(z - 1) = (z - 1)(z^2 + 4) =$
 $= (z - 1)(z - 2i)(z + 2i)$; so $z^3 - z^2 + 4z - 4 = (z - 1)(z - 2i)(z + 2i) = 0$.
 The three roots are $z_1 = 1$, $z_2 = 2i$, $z_3 = -2i$. Since $z_1 = 1 = 1(\cos 0 + i \sin 0)$;
 $z_2 = 2i = 2\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$ and $z_3 = -2i = 2\left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}\right)$ we get:
 $z_1 \cdot z_2 \cdot z_3 = 1 \cdot 2 \cdot 2 \cdot \left(\cos\left(0 + \frac{\pi}{2} + \frac{3\pi}{2}\right) + i \sin\left(0 + \frac{\pi}{2} + \frac{3\pi}{2}\right)\right) =$
 $= 4(\cos 2\pi + i \sin 2\pi) = 4e^{2\pi i} = e^{\log 4 + 2\pi i}$.

I M 2) Since $\mathbb{A} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ m & m & 1 & 1 \end{vmatrix}$, the image of (x_1, x_2, x_3, x_4) can be written

as $F(x_1, x_2, x_3, x_4) = \mathbb{A} \cdot \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix}$; using elementary operations on the rows we get:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ m & m & 1 & 1 \end{vmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \\ m-1 & m-1 & 0 & 0 \end{vmatrix} \text{ from which we easily see}$$

that $\text{Dim}(\text{Imm}) = \begin{cases} 2 & \text{if } m = 1 \\ 3 & \text{otherwise} \end{cases}$ while $\text{Dim}(\text{Ker}) = \begin{cases} 2 & \text{if } m = 1 \\ 1 & \text{otherwise} \end{cases}$; the dimension of the Kernel is maximum if $m = 1$.

For a basis of the Kernel remember that (x_1, x_2, x_3, x_4) belongs to the Kernel of F if

$$\mathbb{A} \cdot \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \mathbb{O}; \text{ in system form we get:}$$

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -x_1 \\ x_4 = -x_2 \end{cases}, \text{ every ele-}$$

ment of the Kernel is $\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} x_1 \\ x_2 \\ -x_1 \\ -x_2 \end{vmatrix} = x_1 \begin{vmatrix} 1 \\ 0 \\ -1 \\ 0 \end{vmatrix} + x_2 \begin{vmatrix} 0 \\ 1 \\ 0 \\ -1 \end{vmatrix}$, so a basis for the

Kernel is $\mathcal{B}_{\text{Ker}(F)} = \{(1, 0, -1, 0), (0, 1, 0, -1)\}$. For a basis of the Image remember

that (y_1, y_2, y_3) belongs to the Image of F if $\mathbb{A} \cdot \begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{vmatrix} = \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix}$, that in system form is

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = y_1 \\ x_1 - x_2 + x_3 - x_4 = y_2 \\ x_1 + x_2 + x_3 + x_4 = y_3 \end{cases} \Rightarrow y_1 = y_3; \text{ so every element of the Image is a vector}$$

(y_1, y_2, y_3) with $y_3 = y_1$ or $\begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} y_1 \\ y_2 \\ y_1 \end{vmatrix} = y_1 \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} + y_2 \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix}$ and a basis for the Image is $\mathcal{B}_{Imm(F)} = \{(1, 0, 1), (0, 1, 0)\}$.

I M 3) The characteristic polynomial of \mathbb{A} is:

$$p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda \mathbb{I}| = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix} = (2 - \lambda)^2 - 1 = \lambda^2 - 4\lambda + 3$$

while the characteristic polynomial of \mathbb{B} is:

$$p_{\mathbb{B}}(\lambda) = |\mathbb{B} - \lambda \mathbb{I}| = \begin{vmatrix} a - \lambda & 0 \\ 0 & b - \lambda \end{vmatrix} = (a - \lambda)(b - \lambda) = \lambda^2 - (a + b)\lambda + ab. \quad \text{The}$$

two matrices have the same characteristic polynomial if and only if $a + b = 4$ and $ab = 3$; so $a = 1$ and $b = 3$ or $a = 3$ and $b = 1$.

For the dimensions of the eigenspaces associated to the eigenvalues of the matrix \mathbb{B} note that the two eigenvalues are simple, i.e. they have algebraic multiplicity equal to 1 and so also their geometric multiplicity is equal to 1. So any eigenspace has dimension equal to 1.

I M 4) A modal matrix that diagonalizes \mathbb{A} , if it exists, is a non singular matrix \mathbb{P} having columns that are linearly independent eigenvectors associated to the matrix \mathbb{A} . To find \mathbb{P} the first step is to calculate the characteristic polynomial of \mathbb{A} :

$$p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda \mathbb{I}| = \begin{vmatrix} 1 - \lambda & 1 & -1 \\ 0 & -\lambda & 0 \\ 2 & 2 & -2 - \lambda \end{vmatrix} = (-\lambda) \begin{vmatrix} 1 - \lambda & -1 \\ 2 & -2 - \lambda \end{vmatrix} =$$

$$= (-\lambda)((1 - \lambda)(-2 - \lambda) + 2) = -\lambda^3 - \lambda^2 = (-\lambda^2)(\lambda + 1).$$

From $-\lambda^2(\lambda + 1) = 0$ we get the three eigenvalues $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = -1$.

Since $m_0^g = 3 - \text{Rank}(\mathbb{A} - 0 \cdot \mathbb{I}) = 3 - \text{Rank}(\mathbb{A}) = 3 - 1 = 2$ we get $m_0^g = m_0^a$ and the matrix is a diagonalizable one.

To find two eigenvectors $v = (v_1, v_2, v_3)$ associated to the eigenvalue 0 we have to solve the system $(\mathbb{A} - 0 \cdot \mathbb{I}) \cdot v = \mathbb{O}$, and so:

$$\begin{cases} v_1 + v_2 - v_3 = 0 \\ 0 = 0 \\ 2v_1 + 2v_2 - 2v_3 = 0 \end{cases} \Rightarrow v_3 = v_1 + v_2, \text{ so every eigenvector associated to 0 is a vec-}$$

$$\text{tor } v = \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} = \begin{vmatrix} v_1 \\ v_2 \\ v_1 + v_2 \end{vmatrix} = v_1 \begin{vmatrix} 1 \\ 0 \\ 1 \end{vmatrix} + v_2 \begin{vmatrix} 0 \\ 1 \\ 1 \end{vmatrix}.$$

To find one eigenvector associated to the eigenvalue -1 we solve the system:

$$(\mathbb{A} - (-1) \cdot \mathbb{I}) \cdot v = \mathbb{O}, \text{ or } \begin{cases} 2v_1 + v_2 - v_3 = 0 \\ v_2 = 0 \\ 2v_1 + 2v_2 - v_3 = 0 \end{cases} \Rightarrow \begin{cases} v_3 = 2v_1 \\ v_2 = 0 \end{cases}, \text{ so every eigenvec-}$$

$$\text{tor associated to } -1 \text{ is a vector } v = \begin{vmatrix} v_1 \\ v_2 \\ v_3 \end{vmatrix} = \begin{vmatrix} v_1 \\ 0 \\ 2v_1 \end{vmatrix} = v_1 \begin{vmatrix} 1 \\ 0 \\ 2 \end{vmatrix}.$$

$$\text{So a modal matrix to diagonalize } \mathbb{A} \text{ may be } P = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}.$$

II M 1) $f(x, y)$ is a differentiable function with $\nabla f = (3x^2, -3y^2)$, so:

$$D_v f(P) = \nabla f \cdot v = (3x^2, -3y^2) \cdot (1, 0) = 3x^2,$$

$$D_w f(P) = \nabla f \cdot w = (3x^2, -3y^2) \cdot (0, 1) = -3y^2.$$

From $D_v f(P) = D_w f(P)$ it follows $3x^2 = -3y^2 \Rightarrow x^2 + y^2 = 0$ whose unique solution is $P = (0, 0)$.

II M 2) The Lagrangian function of the problem is:

$$\Lambda(x, y, z, \lambda) = x - 2y + 4z - \lambda \left(x^2 + y^2 + z^2 - \frac{21}{4} \right).$$

$$I OC: \nabla \Lambda = \left(1 - 2\lambda x, -2 - 2\lambda y, 4 - 2\lambda z, - \left(x^2 + y^2 + z^2 - \frac{21}{4} \right) \right).$$

$$\begin{cases} 1 - 2\lambda x = 0 \\ -2 - 2\lambda y = 0 \\ 4 - 2\lambda z = 0 \\ x^2 + y^2 + z^2 = \frac{21}{4} \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = -\frac{1}{\lambda} \\ z = \frac{2}{\lambda} \\ \frac{21}{4\lambda^2} = \frac{21}{4} \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ y = -1 \\ z = 2 \\ \lambda = 1 \end{cases} \text{ or } \begin{cases} x = -\frac{1}{2} \\ y = 1 \\ z = -2 \\ \lambda = -1 \end{cases}. \text{ The problem}$$

has two constrained critical points: $P_1 = \left(\frac{1}{2}, -1, 2 \right)$ and $P_2 = \left(-\frac{1}{2}, 1, -2 \right)$.

$$II OC: \overline{\mathbb{H}}(\Lambda) = \begin{vmatrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & -2\lambda \end{vmatrix}.$$

Since the problem has three variables and one constraint, we must consider two leading principal minors: $|\overline{\mathbb{H}}_3(\Lambda)|$ and $|\overline{\mathbb{H}}_4(\Lambda)|$.

$$\begin{aligned} |\overline{\mathbb{H}}_3(\Lambda)| &= \begin{vmatrix} 0 & 2x & 2y \\ 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \end{vmatrix} = -2x \begin{vmatrix} 2x & 0 \\ 2y & -2\lambda \end{vmatrix} + 2y \begin{vmatrix} 2x & -2\lambda \\ 2y & 0 \end{vmatrix} = \\ &= 8\lambda x^2 + 8\lambda y^2 = 8\lambda(x^2 + y^2); \end{aligned}$$

$$\begin{aligned} |\overline{\mathbb{H}}_4(\Lambda)| &= \begin{vmatrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & -2\lambda \end{vmatrix} = -2x \begin{vmatrix} 2x & 0 & 0 \\ 2y & -2\lambda & 0 \\ 2z & 0 & -2\lambda \end{vmatrix} + \\ &+ 2y \begin{vmatrix} 2x & -2\lambda & 0 \\ 2y & 0 & 0 \\ 2z & 0 & -2\lambda \end{vmatrix} - 2z \begin{vmatrix} 2x & -2\lambda & 0 \\ 2y & 0 & -2\lambda \\ 2z & 0 & 0 \end{vmatrix} = \\ &= -16\lambda^2 x^2 - 16\lambda^2 y^2 - 16\lambda^2 z^2 = -16\lambda^2(x^2 + y^2 + z^2). \text{ Since:} \end{aligned}$$

$$|\overline{\mathbb{H}}_3(P_1)| > 0; |\overline{\mathbb{H}}_4(P_1)| < 0 \Rightarrow P_1 \text{ is the maximum point with } f(P_1) = \frac{21}{2}$$

$$|\overline{\mathbb{H}}_3(P_2)| < 0; |\overline{\mathbb{H}}_4(P_2)| < 0 \Rightarrow P_2 \text{ is the minimum point with } f(P_2) = -\frac{21}{2}.$$

II M 3) f is a differentiable function with $f(0, 0, 0) = 0$.

$$\nabla f = (e^y + z, xe^y - e^z, -ye^z + x), \nabla f(0, 0, 0) = (1, -1, 0);$$

$$\mathbb{H}(f) = \begin{vmatrix} 0 & e^y & 1 \\ e^y & xe^y & -e^z \\ 1 & -e^z & -ye^z \end{vmatrix}, \mathbb{H}(f(0, 0, 0)) = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix}.$$

The request polynomial is :

$$P_2(x, y, z) = f(0, 0, 0) + \nabla f(0, 0, 0) \cdot (x, y, z) + \frac{1}{2} \|x \ y \ z\| \cdot \mathbb{H}(f(0, 0, 0)) \cdot \begin{vmatrix} x \\ y \\ z \end{vmatrix} =$$

$$= x - y + xy + xz - yz.$$

II M 4) The Jacobian matrix $\mathbb{J} = \frac{\partial(f, g)}{\partial(x, y, z)}$ is:

$$\mathbb{J} = \begin{vmatrix} \sin y + z \sin x + ze^{z(x-y)} & x \cos y - ze^{z(x-y)} & -\cos x + (x-y)e^{z(x-y)} \\ 3x^2y + yz & x^3 - z + xz & -y + xy \end{vmatrix} \quad \text{with}$$

$$\mathbb{J}(P) = \begin{vmatrix} 1 & -1 & -1 \\ 0 & -1 & 0 \end{vmatrix}.$$

$$\text{So } \frac{\partial(y, z)}{\partial(x)} = \mathbb{J}^{-1}(P)_{(y, z)} \cdot \mathbb{J}(P)_{(x)} = \begin{vmatrix} -1 & -1 \\ -1 & 0 \end{vmatrix}^{-1} \cdot \begin{vmatrix} 1 \\ 0 \end{vmatrix} =$$

$$= - \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \cdot \begin{vmatrix} 1 \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ -1 \end{vmatrix}.$$

$$\text{Or } \frac{dy}{dx} = - \frac{\begin{vmatrix} 1 & -1 \\ 0 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & -1 \\ -1 & 0 \end{vmatrix}} = 0; \quad \frac{dz}{dx} = - \frac{\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix}}{\begin{vmatrix} -1 & -1 \\ -1 & 0 \end{vmatrix}} = -1.$$