TASK MATHEMATICS for ECONOMIC APPLICATIONS
06/09/2017

I M 1) Since $z = -27i^3 = 27i$, to find the three cubic roots of z we remember that $i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$. So $\sqrt[3]{z} = \sqrt[3]{27i} = 3\sqrt[3]{i} =$

$= 3\Big(\cos\left(\frac{\pi}{6} + \frac{2}{3}k\pi\right) + i\sin\left(\frac{\pi}{6} + \frac{2}{3}k\pi\right)\Big)$ with $k = 0, 1, 2$. The three roots are:

$z_1 = 3\Big(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\Big) = \frac{3}{2}\sqrt{3} + \frac{3}{2}i = \frac{3}{2}\big(\sqrt{3} + i\big)$,

$z_2 = 3\Big(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\Big) = -\frac{3}{2}\sqrt{3} + \frac{3}{2}i = \frac{3}{2}\big(-\sqrt{3} + i\big)$ and

$z_3 = 3\Big(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\Big) = -3i$.

I M 2) The characteristic polynomial of $\mathbb{A}$ is

$$p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & -9 & -3-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 4-\lambda & 2 \\ 4+\lambda & -9 & -3-\lambda \end{vmatrix} =$$

$$= (2-\lambda)\begin{vmatrix} 4-\lambda & 2 \\ -9 & -3-\lambda \end{vmatrix} + (4+\lambda)\begin{vmatrix} 1 & 1 \\ 4-\lambda & 2 \end{vmatrix} =$$

$$= (2-\lambda)\big(\lambda^2 - \lambda + 6\big) + (4+\lambda)(\lambda - 2) = (2-\lambda)\big(\lambda^2 - 2\lambda + 2\big).$$

Since $\lambda^2 - 2\lambda + 2 = 0$ for $\lambda = 1 \pm i$, the matrix $\mathbb{A}$ admits the three eigenvalues $\lambda_1 = 2$, $\lambda_2 = 1 - i$ and $\lambda_3 = 1 + i$ and it is diagonalizable because it admits three distinct eingenvalues.

I M 3) By conditions proposed on basis $\mathbb{W}$ and vector $\mathbb{X}$ it follows:

$$\mathbb{X} = \begin{Vmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 2 & 1 & k \end{Vmatrix} \cdot \begin{Vmatrix} 2 \\ -1 \\ 1 \end{Vmatrix} = 2(1, -1, 2) - (2, 1, 1) + (1, 2, k) = (1, -1, 3 + k).$$

It is easy to find $k = -1$. So $\mathbb{X} = (1, -1, 2)$ and $||\mathbb{X}|| = \sqrt{1 + 1 + 4} = \sqrt{6}$.

I M 4) Since the non-singular matrices $\mathbb{A}$ and $\mathbb{B}$ have the eigenvector $\mathbb{X}$ corresponding to the eigenvalue λ, it follows $\mathbb{A} \cdot \mathbb{X} = \lambda\,\mathbb{X}$ and $\mathbb{B} \cdot \mathbb{X} = \lambda\,\mathbb{X}$, with $\lambda \neq 0$.

Remember that for the inverse matrix $\mathbb{A}^{-1}$ we have $\mathbb{A}^{-1} \cdot \mathbb{X} = \frac{1}{\lambda}\mathbb{X}$. And so

$$\mathbb{M} \cdot \mathbb{X} = \big(\mathbb{A}^2 \cdot \mathbb{B}^{-1} + 3\,\mathbb{B} \cdot \mathbb{A}^{-1} \cdot \mathbb{B}\big) \cdot \mathbb{X} = \mathbb{A}^2 \cdot \mathbb{B}^{-1} \cdot \mathbb{X} + 3\,\mathbb{B} \cdot \mathbb{A}^{-1} \cdot \mathbb{B} \cdot \mathbb{X} =$$

$$= \mathbb{A} \cdot \mathbb{A} \cdot \mathbb{B}^{-1} \cdot \mathbb{X} + 3\,\mathbb{B} \cdot \mathbb{A}^{-1} \cdot \mathbb{B} \cdot \mathbb{X} = \mathbb{A} \cdot \mathbb{A} \cdot \frac{1}{\lambda}\mathbb{X} + 3\,\mathbb{B} \cdot \mathbb{A}^{-1} \cdot \lambda\,\mathbb{X} =$$

$$= \frac{1}{\lambda}\mathbb{A} \cdot \mathbb{A} \cdot \mathbb{X} + 3\lambda\,\mathbb{B} \cdot \mathbb{A}^{-1} \cdot \mathbb{X} = \frac{1}{\lambda}\mathbb{A} \cdot \lambda\,\mathbb{X} + 3\lambda\,\mathbb{B} \cdot \frac{1}{\lambda}\mathbb{X} =$$

$$= \mathbb{A} \cdot \mathbb{X} + 3\,\mathbb{B} \cdot \mathbb{X} = \lambda\,\mathbb{X} + 3\,\lambda\,\mathbb{X} = 4\lambda\,\mathbb{X}.$$

The eigenvalue of matrix $\mathbb{M}$ that corresponds to eigenvector $\mathbb{X}$ is 4λ.

II M 1) Since $|\mathbb{A}(x, y)| = \begin{vmatrix} x & y \\ x^2 & e^y \end{vmatrix} = xe^y - x^2y$, the problem is equivalent to find

$\text{Max/min}\, f(x, y) = xe^y - x^2y$.

$I\,OC$: $\nabla|\mathbb{A}(x,y)| = \left(e^y - 2xy, xe^y - x^2\right)$. $\begin{cases} e^y - 2xy = 0 \\ xe^y - x^2 = 0 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} e^y - 2xy = 0 \\ x(e^y - x) = 0 \end{cases} \begin{matrix} \nearrow & \begin{cases} e^y = 0 \\ x = 0 \end{cases} \text{; system impossible.} \\ \searrow & \begin{cases} e^y - 2e^y y = 0 \\ x = e^y \end{cases} \Rightarrow \begin{cases} e^y(1-2y) = 0 \\ x = e^y \end{cases} \Rightarrow \begin{cases} y = 1/2 \\ x = \sqrt{e} \end{cases} \end{matrix}.$$

We get only one stationary point $P = \left(\sqrt{e}, 1/2\right)$.

$\mathbb{H}(f(x,y)) = \begin{Vmatrix} -2y & e^y - 2x \\ e^y - 2x & xe^y \end{Vmatrix}$; $|\mathbb{H}(f(x,y))| = -2xye^y - (e^y - 2x)^2$.

$II\,OC$: $|\mathbb{H}(f(P))| = -e - \left(-\sqrt{e}\right)^2 = -2e < 0$. So P is a saddle point. There are no values (x,y) which maximise or minimize the determinant of the matrix $\mathbb{A}$.

II M 2) Problem $\begin{cases} \text{Max/min } f(x,y) = x^2 y \\ \text{u.c. } x^2 + y^2 \le 1 \end{cases}$ is equivalent to the problem:

$\begin{cases} \text{Max/min } f(x,y) = x^2 y \\ \text{u.c. } x^2 + y^2 - 1 \le 0 \end{cases}$; the admissible region is drawed in red in the figure in the next page; the objective function is continuous, the admissible region is bounded and closed, so by Weierstrass Theorem the problem admits the absolute maximum and minimum values.

The Lagrangian function of the problem is: $\Lambda(x,y,\lambda) = x^2y - \lambda\left(x^2 + y^2 - 1\right)$

whose gradient is: $\nabla\Lambda(x,y,\lambda) = \left(2xy - 2\lambda x, x^2 - 2\lambda y, -\left(x^2 + y^2 - 1\right)\right)$.

KUHN-TUCKER CONDITIONS

First case (*free optimization*): $\begin{cases} \lambda = 0 \\ 2xy = 0 \\ x^2 = 0 \\ x^2 + y^2 - 1 \le 0 \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ 0 = 0 \\ x = 0 \\ y^2 \le 1 \end{cases}$, the system is satisfied for every point $\overline{P} = (0, \overline{y})$ with $-1 \le \overline{y} \le 1$.

$\mathbb{H}(f) = \begin{Vmatrix} 2y & 2x \\ 2x & 0 \end{Vmatrix}$, $\mathbb{H}\left(f\left(\overline{P}\right)\right) = \begin{Vmatrix} 2\overline{y} & 0 \\ 0 & 0 \end{Vmatrix}$ and $\left|\mathbb{H}\left(f\left(\overline{P}\right)\right)\right| = 0$; we can't conclude anything about the nature of the points $\overline{P}$ by the hessian of f.

If we consider the difference $f(x,y) - f(0,\overline{y}) = x^2 y$ in a neighbourhood of $\overline{P}$, we have that $f(x,y) - f(0,\overline{y})$ is $\begin{pmatrix} < 0 \\ = 0 \\ > 0 \end{pmatrix}$ if $\begin{pmatrix} y < 0 \text{ and } x \ne 0 \\ y = 0 \text{ and } x = 0 \\ y > 0 \text{ and } x \ne 0 \end{pmatrix}$ so we conclude that if $-1 \le \overline{y} < 0$, $\overline{P}$ is a local maximum point, if $0 < \overline{y} \le 1$, $\overline{P}$ is a local minimum point, while $(0,0)$ is a saddle point.

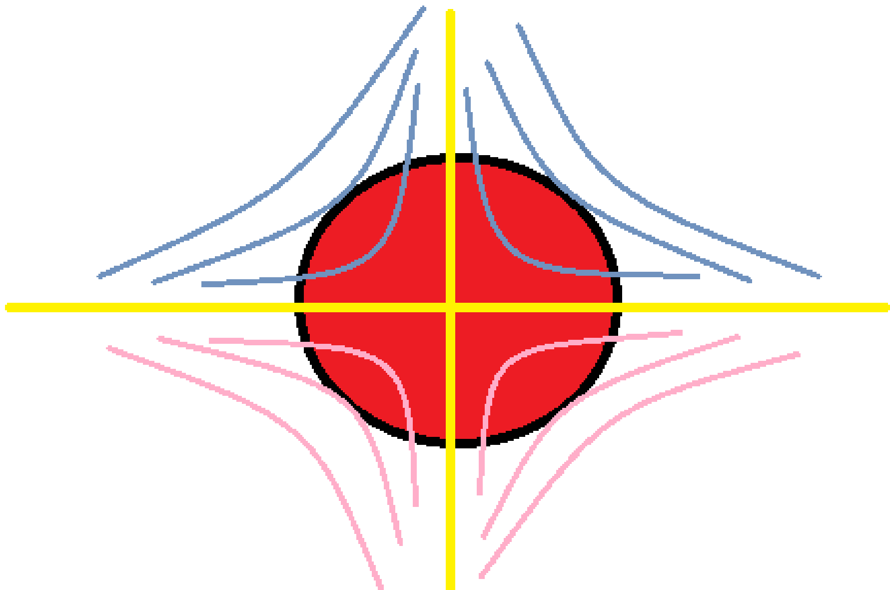
Second case (*constraint is active*): $\begin{cases} \lambda \ne 0 \\ 2xy - 2\lambda x = 0 \\ x^2 - 2\lambda y = 0 \\ x^2 + y^2 - 1 = 0 \end{cases} \Rightarrow$

$$\begin{cases} \lambda \neq 0 \\ 2x(y-\lambda)=0 \\ x^2-2\lambda y=0 \\ x^2+y^2=1 \end{cases} \begin{matrix} \nearrow \begin{cases} \lambda \neq 0 \\ x=0 \\ -2\lambda y=0 \\ y^2=1 \end{cases} \Rightarrow \begin{matrix} \lambda \neq 0 \\ x=0 \\ \lambda=0 \\ y=\pm 1 \end{matrix} \text{; the system is impossible.} \\ \searrow \begin{cases} \lambda \neq 0 \\ y=\lambda \\ x^2=2\lambda^2 \\ 3\lambda^2=1 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ y=\pm\sqrt{3}/3 \\ x^2=2/3 \\ \lambda=\pm\sqrt{3}/3 \end{cases} \Rightarrow \begin{cases} \lambda \neq 0 \\ y=\pm\sqrt{3}/3 \\ x=(\pm)\sqrt{6}/3 \\ \lambda=\pm\sqrt{3}/3 \end{cases} \end{matrix}$$

. We get four constrained critical points: $P_{1,2}=\left(\pm\sqrt{6}/3, \sqrt{3}/3, \sqrt{3}/3\right)$, possible maximum points since $\lambda > 0$ and $P_{3,4}=\left(\pm\sqrt{6}/3, -\sqrt{3}/3, -\sqrt{3}/3\right)$ possible minimum points since $\lambda < 0$. Furthermore $f(P_{1,2})=2\sqrt{3}/9$, $f(P_{3,4})=-2\sqrt{3}/9$. So:

$$\text{Max}(f)=f\left(\pm\frac{\sqrt{6}}{3}, \frac{\sqrt{3}}{3}\right)=\frac{2\sqrt{3}}{9} \text{ ; } \min(f)=f\left(\pm\frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}\right)=-\frac{2\sqrt{3}}{9}.$$

For constraints qualification consider its Jacobian: $\mathbb{J}(g)=(2x,2y)$ and $\mathbb{J}(g)=(0,0)$ only at point $(0,0)$, point not belonging to the constraint. So the constrait is qualified.
On the figure below there are drawn zero level curves (yellow), positive level curves (blue) and negative level curves (pink).



II M 3) Since $||\mathbb{X}_1||=\sqrt{1^2+1^2}=\sqrt{2}$ and $||\mathbb{X}_2||=\sqrt{1^2+(-1)^2}=\sqrt{2}$, thus

$$u=\frac{1}{||\mathbb{X}_1||}\cdot\mathbb{X}_1=\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ and } v=\frac{1}{||\mathbb{X}_2||}\cdot\mathbb{X}_2=\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right).$$

$f(x,y)$ is a twice differentiable function and so, from $\nabla f=(2x-2y,\ 1-2x)$ it follows:

$$\mathcal{D}_u f(P)=\nabla f\cdot u=(2x-2y,\ 1-2x)\cdot\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)=\frac{\sqrt{2}}{2}(1-2y) \text{ and}$$

$$\mathcal{D}_v f(P) = \nabla f \cdot v = (2x - 2y,\ 1 - 2x) \cdot \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2}}{2}(4x - 2y - 1).$$

From the given conditions we get:

$$\begin{cases} \mathcal{D}_u f(P) = 0 \\ \mathcal{D}_v f(P) = \sqrt{2} \end{cases} \Rightarrow \begin{cases} \frac{\sqrt{2}}{2}(1 - 2y) = 0 \\ \frac{\sqrt{2}}{2}(4x - 2y - 1) = \sqrt{2} \end{cases} \Rightarrow \begin{cases} 1 - 2y = 0 \\ 4x - 2y - 1 = 2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 1 \\ y = 1/2 \end{cases},\ P = \left(1, \frac{1}{2}\right).\ \mathbb{H}(f) = \begin{Vmatrix} 2 & -2 \\ -2 & 0 \end{Vmatrix},\ \mathcal{D}^2_{u,v} f(P) = u \cdot \mathbb{H}(f) \cdot v^T =$$

$$= \begin{Vmatrix} \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \end{Vmatrix} \cdot \begin{Vmatrix} 2 & -2 \\ -2 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{Vmatrix} = \begin{Vmatrix} \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \end{Vmatrix} \cdot \begin{Vmatrix} 2\sqrt{2} \\ -\sqrt{2} \end{Vmatrix} = 1.$$

II M 4) Let us calculate the gradiente of $f(x, y)$:

$$\nabla f = \left(\log\left(1 + x^2 + y^2\right) + \frac{(x+y) \cdot 2x}{1 + x^2 + y^2},\ \log\left(1 + x^2 + y^2\right) + \frac{(x+y) \cdot 2y}{1 + x^2 + y^2} \right),$$

$$\nabla f(P) = (\log 3, \log 3)\,;\ y'(1) = -\frac{f'_x(P)}{f'_y(P)} = -\frac{\log 3}{\log 3} = -1.$$