TASK MATHEMATICS for ECONOMIC APPLICATIONS
06/09/2017

I M 1) $w=\dfrac{i-\overline{z}}{i+z}=\dfrac{i-(2-i)}{i+(2+i)}=\dfrac{2i-2}{2i+2}=\dfrac{i-1}{i+1}\cdot\dfrac{1-i}{1-i}=\dfrac{2i}{2}=i$ .To find the two square roots of w remember that $i=\cos\left(\dfrac{\pi}{2}\right)+i\sin\left(\dfrac{\pi}{2}\right)$, thus

$$\sqrt{i}=\sqrt{\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)}=\left(\cos\left(\frac{\pi}{4}+k\pi\right)+i\sin\left(\frac{\pi}{4}+k\pi\right)\right)\quad k=0,1\,.$$

The two roots are:

$$z_1=\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i=\frac{\sqrt{2}}{2}(1+i)\ \text{ and}$$
$$z_2=\cos\left(\frac{5}{4}\pi\right)+i\sin\left(\frac{5}{4}\pi\right)=-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i=-\frac{\sqrt{2}}{2}(1+i)=-z_1\,.$$

I M 2) From $F(x_1,x_2,x_3,x_4)=(x_1+x_2+x_3+x_4,mx_1+x_2+x_3+mx_4)$ we get: $F(\mathbb{X})=\mathbb{A}\cdot\mathbb{X}$ with $\mathbb{A}=\begin{Vmatrix}1&1&1&1\\m&1&1&m\end{Vmatrix}$. From $F(1,1,1,1)=(4,2m+2)$ easily we get $m=1$. Since the dimension of the Image of a linear map is equal to the rank of the matrix $\mathbb{A}$ associated to the map, for the matrix $\mathbb{A}=\begin{Vmatrix}1&1&1&1\\1&1&1&1\end{Vmatrix}$, by elementary operations on the rows we get: $\begin{Vmatrix}1&1&1&1\\1&1&1&1\end{Vmatrix}\overset{R_2-R_1}{\longrightarrow}\begin{Vmatrix}1&1&1&1\\0&0&0&0\end{Vmatrix}$ and so we easily get $\text{Rank}(\mathbb{A})=1=\text{Dim}(\text{Imm})$.

From $\text{Dim}(\text{Imm})=1$ we get $\text{Dim}(\text{Ker})=4-1=3$. Since the vector $(4,4)$ belongs to the Image of F, a basis for $\text{Imm}(F)$ is $\mathcal{B}_{Imm(F)}=\{(1,1)\}$. To find a basis for the Kernel we solve the system: $\begin{cases}x_1+x_2+x_3+x_4=0\\x_1+x_2+x_3+x_4=0\end{cases}\Rightarrow x_4=-x_1-x_2-x_3$.

Every element $\mathbb{X}\in\text{Ker}$ is $\mathbb{X}=(x_1,x_2,x_3,-x_1-x_2-x_3)$ and a basis for the Kernel is $\mathcal{B}_{Ker(F)}=\{(1,0,0,-1),(0,1,0,-1),(0,0,1,-1)\}$.

I M 3) The characteristic polynomial of $\mathbb{A}$ is $p_{\mathbb{A}}(\lambda)=|\mathbb{A}-\lambda\mathbb{I}|=$

$$=\begin{vmatrix}2-\lambda&2&1\\0&1-\lambda&-3\\1&2&k-\lambda\end{vmatrix}\overset{R_1-R_3}{=}\begin{vmatrix}1-\lambda&0&1+\lambda-k\\0&1-\lambda&-3\\1&2&k-\lambda\end{vmatrix}=$$
$$=(1-\lambda)\begin{vmatrix}1-\lambda&-3\\2&k-\lambda\end{vmatrix}+(1+\lambda-k)\begin{vmatrix}0&1-\lambda\\1&2\end{vmatrix}=$$
$$=(1-\lambda)\big(\lambda^2-\lambda-k\lambda+k\big)+(1+\lambda-k)(\lambda-1)=$$
$$=(1-\lambda)\big(\lambda^2-(2+k)\lambda+2k-1\big)\,.$$

So $\lambda=1$ is an eigenvalue for the matrix $\mathbb{A}$ for every value of the parameter k.
To have multiple eigenvalues for $\mathbb{A}$ there are two possibilities:
a) $\lambda=1$ is a root also for $\lambda^2-(2+k)\lambda+2k-1=0$;
b) $\lambda^2-(2+k)\lambda+2k-1=0$ has a multiple (double) root.
a) $\lambda=1$ is a root for $\lambda^2-(2+k)\lambda+2k-1=0$ if $1-2-k+2k-1=0$, satisfied for $k=2$. If $k=2$ we get $\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)=0$ with roots $\lambda=1$ and $\lambda=3$. For $\lambda=1$ and $k=2$ we have $\|\mathbb{A}-1\cdot\mathbb{I}\|=\begin{Vmatrix}1&2&1\\0&0&-3\\1&2&1\end{Vmatrix}$ whose Rank

is clearly equal to 2 and so $m_1^g = 3 - 2 = 1 < m_1^a = 2$. For $k = 2$ the matrix is not a diagonalizable one.

b) $\lambda^2 - (2+k)\lambda + 2k - 1 = 0 \Rightarrow \lambda = \dfrac{(2+k) \pm \sqrt{(2+k)^2 - 4(2k-1)}}{2}$ and we have a multiple (double) root if $(2+k)^2 - 4(2k-1) = k^2 - 4k + 8 = 0$. But this equation does not have real solutions and therefore case b) does not provide multiple solutions. So the matrix $\mathbb{A}$ is not diagonalizable only for $k = 2$.

I M 4) Since $\mathbb{Y} = \mathbb{A} \cdot \mathbb{X} = \begin{Vmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ k \\ 0 \end{Vmatrix} = \begin{Vmatrix} 1-k \\ 1+2k \\ 1-k \end{Vmatrix}$ we get:
$||\mathbb{Y}|| = \sqrt{(1-k)^2 + (1+2k)^2 + (1-k)^2} = \sqrt{6k^2+3}$. $\mathbb{Y}$ has modulus equal to 3 if and only if $6k^2 + 3 = 9$ or $k = \pm 1$.

II M 1) $\nabla f = \big(2x + y^2, -2y + 2xy\big)$. $\mathbb{H}(x,y) = \begin{Vmatrix} 2 & 2y \\ 2y & -2+2x \end{Vmatrix}$.

$I\,OC$: $\begin{cases} 2x + y^2 = 0 \\ 2xy - 2y = 2y(x-1) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \cup \begin{cases} y^2 = -2 \\ x = 1 \end{cases}$ Impossible.

We have only the stationary point $(0,0)$.

$II\,OC$: $\mathbb{H}(0,0) = \begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix}$. Since $f''_{xx} > 0$ while $f''_{yy} < 0$ the point $(0,0)$ is a saddle point.

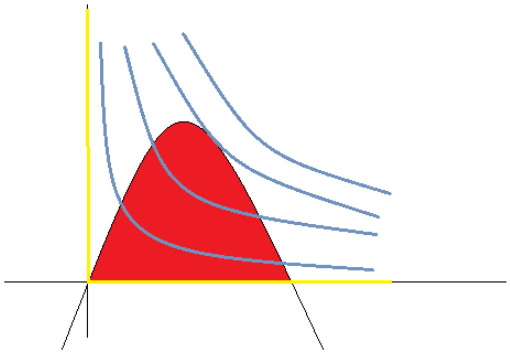
II M 2) The problem $\begin{cases} \text{Max/min } f(x,y) = xy \\ \text{u.c. } 0 \le y \le 2x - x^2 \end{cases} \Rightarrow \begin{cases} \text{Max/min } f(x,y) = xy \\ \text{u.c. } \begin{cases} -y \le 0 \\ y - 2x + x^2 \le 0 \end{cases} \end{cases}$.

The feasible region is red-drawed in the figure in the next page.
The constraints are qualified, the objective function is continous and the feasible region is a bounded and closed set, so by Weierstrass Theorem the problem admits the absolute maximum and minimum value.
In the feasible region we have $f(x,y) \ge 0$ with $f(x,y) = 0$ only for $y = 0$.
So all the points $\{0 \le x \le 2, y = 0\}$ are minimum points.
The function $f(x,y) = xy$ has a unique stationary point: $(0,0)$, just studied.
On the points satisfying $y - 2x + x^2 = 0$ we apply the Kuhn-Tucker condition using the Lagrangian function $\Lambda(x,y,\lambda) = xy - \lambda\big(y - 2x + x^2\big)$.
We have to solve the system:

$$\begin{cases} y + 2\lambda - 2\lambda x = 0 \\ x - \lambda = 0 \\ y = 2x - x^2 \end{cases} \Rightarrow \begin{cases} \lambda = x \\ y = 2x^2 - 2x \\ 2x^2 - 2x = 2x - x^2 \end{cases} \Rightarrow \begin{cases} \lambda = x \\ y = 2x^2 - 2x \\ 3x^2 - 4x = x(3x-4) = 0 \end{cases}$$

and so we get the two solutions: $\begin{cases} x = 0 \\ y = 0 \\ \lambda = 0 \end{cases}$ just studied and $\begin{cases} x = \frac{4}{3} \\ y = \frac{8}{9} \\ \lambda = \frac{4}{3} \end{cases}$ which is, clearly, the maximum point.

We conclude: $\text{Max}(f) = f\left(\frac{4}{3}, \frac{8}{9}\right) = \frac{32}{27}$; $\text{Min}(f) = f(\overline{x}, 0) = 0, 0 \le \overline{x} \le 2$.
In the figure below are drawn zero level curve (yellow) and positive level curves (blue).



II M 3) $f(x,y) = e^{x-y} - e^{y-x}$ is a differentiable function, so $D_v f(P) = \nabla f(P) \cdot v$ and $D^2_{v,w} f(P) = v \cdot \mathbb{H}(P) \cdot w^T$.
$\nabla f = (e^{x-y} + e^{y-x},\ -e^{x-y} - e^{y-x})$, $\nabla f(1,1) = (2, -2)$, and so:
$D_v f(1,1) = \nabla f(1,1) \cdot v = (2, -2) \cdot (\cos\alpha, \operatorname{sen}\alpha) = 2(\cos\alpha - \operatorname{sen}\alpha)$.
$D_v f(1,1) = 0$ if and only if $\cos\alpha = \operatorname{sen}\alpha$ true for $\alpha = \frac{\pi}{4}$ and $\alpha = \frac{3}{4}\pi$.
Given $w = (\cos\beta, \operatorname{sen}\beta)$, since $\mathbb{H}(x,y) = \begin{Vmatrix} e^{x-y} - e^{y-x} & -e^{x-y} + e^{y-x} \\ -e^{x-y} + e^{y-x} & e^{x-y} - e^{y-x} \end{Vmatrix}$, it results $\mathbb{H}(1,1) = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}$ and finally $D^2_{v,w} f(1,1) = v \cdot \mathbb{H}(1,1) \cdot w^T = 0$, independently from w.

II M 4) The system $\begin{cases} f(x,y,z) = x^2 y - e^{x-z} + e^{z-y} = 1 \\ g(x,y,z) = xyz + e^{x-z} - e^{z-y} = 1 \end{cases}$ at point $(1,1,1,)$ is satisfied. From $\mathbb{J}(x,y,z) = \frac{\partial(f,g)}{\partial(x,y,z)} = \begin{Vmatrix} 2xy - e^{x-z} & x^2 - e^{z-y} & e^{x-z} + e^{z-y} \\ yz + e^{x-z} & xz + e^{z-y} & xy - e^{x-z} - e^{z-y} \end{Vmatrix}$ we get $\mathbb{J}(1,1,1) = \begin{Vmatrix} 1 & 0 & 2 \\ 2 & 2 & -1 \end{Vmatrix}$. Since $\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} = 2 \neq 0$ with this system we can define, in a neighborhood of the point $(1,1,1)$, an implicit function $z \longrightarrow (x(z), y(z))$.
For this function we get:

$$\frac{\mathrm{d}x}{\mathrm{d}z}(1) = x'(1) = -\frac{\begin{vmatrix} 2 & 0 \\ -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}} = -\frac{4}{2} = -2 \text{ and}$$

$$\frac{\mathrm{d}y}{\mathrm{d}z}(1) = y'(1) = -\frac{\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix}} = -\frac{-5}{2} = \frac{5}{2} .$$

The equation of the tangent line at $z = 1$ is $z \longrightarrow (x(1), y(1)) + z\,(x'(1), y'(1))$ or $z \longrightarrow (1,1) + z\left(-2, \frac{5}{2}\right) = \left(1 - 2z, 1 + \frac{5}{2}z\right)$, or, in cartesian form:
$y = -\frac{5}{4}x + \frac{9}{4}$.