INTERMEDIATE TEST

MATHEMATICS for ECONOMIC APPLICATIONS 04/12/2017

I M 1) Using trigonometric form: $1+i=\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\,\text{sen}\left(\frac{\pi}{4}\right)\right)=\sqrt{2}\,e^{i\frac{\pi}{4}}$ and $1+\sqrt{3}\,i=2\left(\cos\left(\frac{\pi}{3}\right)+i\,\text{sen}\left(\frac{\pi}{3}\right)\right)=2\,e^{i\frac{\pi}{3}}$; thus $\dfrac{1+i}{1+\sqrt{3}\,i}=\dfrac{\sqrt{2}\,e^{i\frac{\pi}{4}}}{2\,e^{i\frac{\pi}{3}}}=\dfrac{\sqrt{2}}{2}e^{-i\frac{\pi}{12}}$ and $\left(\dfrac{1+i}{1+\sqrt{3}\,i}\right)^4=\left(\dfrac{\sqrt{2}}{2}e^{-i\frac{\pi}{12}}\right)^4=\dfrac{1}{4}e^{-i\frac{\pi}{3}}=\dfrac{1}{4}\left(\cos\left(-\frac{\pi}{3}\right)+i\,\text{sen}\left(-\frac{\pi}{3}\right)\right)=$

$$=\frac{1}{4}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=\frac{1}{8}-\frac{\sqrt{3}}{8}i\,.$$

I M 2) The characteristic polynomial of $\mathbb{A}$ is $p_{\mathbb{A}}(\lambda)=\begin{vmatrix}2-\lambda & 2 & 1\\ 0 & k-\lambda & 0\\ 1 & 2 & 2-\lambda\end{vmatrix}=$

$=(k-\lambda)\begin{vmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix}=(k-\lambda)\left((2-\lambda)^2-1\right)=(k-\lambda)\left(\lambda^2-4\lambda+3\right)=$

$=(k-\lambda)(\lambda-3)(\lambda-1)$; the matrix $\mathbb{A}$ has multiple eigenvalue if $k=1$ or $k=3$. For $k=1$ the multiple eigenvalue is $\lambda=1$ with algebraic multiplicity equal to 2.

The matrix $\|\mathbb{A}-1\cdot\mathbb{I}\|=\left\|\begin{matrix}1 & 2 & 1\\ 0 & 0 & 0\\ 1 & 2 & 1\end{matrix}\right\|$ has rank equal to 1, so the geometric multiplicity of $\lambda=1$ is equal to 2 and so for $k=1$ the matrix is diagonalizable.

For $k=3$ the multiple eigenvalue is $\lambda=3$ has algebraic multiplicity equal to 2 but the matrix $\|\mathbb{A}-3\cdot\mathbb{I}\|=\left\|\begin{matrix}-1 & 2 & 1\\ 0 & 0 & 0\\ 1 & 2 & -1\end{matrix}\right\|$ has rank equal to 2, so its geometric multiplicity is equal to 1 and so for $k=3$ the matrix is not diagonalizable.

I M 3) The dimension of the Image of a linear map is equal to the rank of the matrix associated to the map; the matrix $\mathbb{A}$, using the elementary operations:

$\mathbb{R}_1\leftrightarrow\mathbb{R}_4$, $\mathbb{R}_2\leftrightarrow\mathbb{R}_3$, $\mathbb{R}_4\leftarrow\mathbb{R}_4-m\cdot\mathbb{R}_1$, can be reduced to $\left\|\begin{matrix}1 & 0 & 0 & k\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1-mk\end{matrix}\right\|$

and easily we see that: $\text{Rank}(\mathbb{A})=\begin{cases}3 & \text{if } km=1\\ 4 & \text{if } km\neq 1\end{cases}$.

So if $km=1$ $\text{Dim}(\text{Imm}(f))=3$ and $\text{Dim}(\text{Ker}(f))=1$ while $\text{Dim}(\text{Imm}(f))=4$ and $\text{Dim}(\text{Ker}(f))=0$ if $km\neq 1$.

I M 4) An element $\mathbb{X}$ belongs to the Kernel of the linear map if $f(\mathbb{X})=\mathbb{A}^T\cdot\mathbb{X}=\mathbb{O}$ or

$\left\|\begin{matrix}1 & 1 & 2\\ 2 & 1 & 1\\ 3 & 2 & 3\\ 1 & 0 & -1\end{matrix}\right\|\cdot\left\|\begin{matrix}x_1\\ x_2\\ x_3\end{matrix}\right\|=\left\|\begin{matrix}0\\ 0\\ 0\\ 0\end{matrix}\right\|$ that in system form is:

$$\begin{cases} x_1 + x_2 + 2x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \\ 3x_1 + 2x_2 + 3x_3 = 0 \\ x_1 - x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_2 + 3x_3 = 0 \\ x_2 + 3x_3 = 0 \\ 2x_2 + 6x_3 = 0 \\ x_1 = x_3 \end{cases} \Rightarrow \begin{cases} x_2 = -3x_3 \\ x_1 = x_3 \end{cases}.$$

Every element of the Kernel is $\begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} x_3 \\ -3x_3 \\ x_3 \end{Vmatrix} = x_3 \cdot \begin{Vmatrix} 1 \\ -3 \\ 1 \end{Vmatrix}$ and so a basis for the Kernel is $\mathcal{B}_{Kerf} = \{(1, -3, 1)\}$.

I M 5) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{Vmatrix}$ we must verify that it has two eigenvalues equal to those of the matrix $\mathbb{A}^2 = \mathbb{A} \cdot \mathbb{A}$.

$$\mathbb{A}^2 = \mathbb{A} \cdot \mathbb{A} = \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{Vmatrix} = \begin{Vmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{Vmatrix}.$$

The characteristic polynomial of $\mathbb{A}$ is:

$$p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\big((1-\lambda)^2 - 1\big) = (1-\lambda)\big(\lambda^2 - 2\lambda\big) = \lambda(1-\lambda)(\lambda-2) = 0$$

and so the three eigenvalues of $\mathbb{A}$ are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 2$.

The characteristic polynomial of $\mathbb{A}^2$ is:

$$p_{\mathbb{A}^2}(\lambda) = \left|\mathbb{A}^2 - \lambda\mathbb{I}\right| = \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} =$$

$$= (1-\lambda)\big((2-\lambda)^2 - 4\big) = (1-\lambda)\big(\lambda^2 - 4\lambda\big) = \lambda(1-\lambda)(\lambda-4) = 0$$

and so the three eigenvalues of $\mathbb{A}^2$ are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 4$.

The Matrices $\mathbb{A}$ and $\mathbb{A}^2$ have two common eigenvalues: $\lambda = 0$ and $\lambda = 1$.