

IM1) $z = e^{\log 16 - i\pi} = e^{\log 16} \cdot e^{-i\pi} = 16 \cdot (\cos(-\pi) + i \sin(-\pi)) = 16(\cos \pi + i \sin \pi) = -16.$

$$\sqrt[n]{z} = \sqrt[n]{16} \cdot \left(\cos \left(\frac{\pi}{n} + K \frac{2\pi}{n} \right) + i \sin \left(\frac{\pi}{n} + K \frac{2\pi}{n} \right) \right); 0 \leq K \leq 3.$$

Per $K=0$: $2 \left(\cos \frac{\pi}{n} + i \sin \frac{\pi}{n} \right) = \sqrt[4]{2} + i \sqrt[4]{2}$; Per $K=1$: $2 \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = -\sqrt[4]{2} + i \sqrt[4]{2}$;

Per $K=2$: $2 \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right) = -\sqrt[4]{2} - i \sqrt[4]{2}$; Per $K=3$: $2 \left(\cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) = \sqrt[4]{2} - i \sqrt[4]{2}$.

IM2) $f(x; y) = \begin{cases} \frac{(x^2 y^2)^\alpha}{(x^2 + y^2)^\alpha} & : (x; y) \neq (0; 0) \\ 0 & : (x; y) = (0; 0) \end{cases}$. Continuità: $\lim_{(x; y) \rightarrow (0; 0)} \frac{(x^2 y^2)^\alpha}{(x^2 + y^2)^\alpha} = ? = 0 \Rightarrow$

$$\Rightarrow \lim_{\rho \rightarrow 0} \frac{\rho^{4\alpha} \cos^2 \vartheta \sin^2 \vartheta}{\rho^{2\alpha}} = \lim_{\rho \rightarrow 0} \rho^{2\alpha} (\cos^2 \vartheta \cdot \sin^2 \vartheta) = 0. \text{ Dato che}$$

$|\rho^{2\alpha} \cdot \cos^2 \vartheta \cdot \sin^2 \vartheta - 0| \leq \rho^{2\alpha} \cdot 1 < \varepsilon$ la convergenza è uniforme e quindi la funzione è continua anche in $(0; 0)$. Calcoliamo le derivate parziali in $(0; 0)$.

$$\frac{\partial f}{\partial x}(0; 0) = \lim_{h \rightarrow 0} \left(\frac{(h^2 \cdot 0)^\alpha}{(h^2 + 0)^\alpha} - 0 \right) \cdot \frac{1}{h} = \lim_{h \rightarrow 0} 0 = 0 = \frac{\partial f}{\partial y}(0; 0). \text{ Occorre calcolare:}$$

$$\lim_{(x; y) \rightarrow (0; 0)} \left(\frac{(x^2 y^2)^\alpha}{(x^2 + y^2)^\alpha} - 0 - (0; 0)(x-0; y-0) \right) \cdot \frac{1}{\sqrt{x^2 + y^2}} = ? = 0 \Rightarrow$$

$$\Rightarrow \lim_{\rho \rightarrow 0} \frac{\rho^{4\alpha} \cos^2 \vartheta \sin^2 \vartheta}{\rho^{2\alpha}} \cdot \frac{1}{\rho} = \lim_{\rho \rightarrow 0} \rho^{2\alpha-1} \cdot \cos^2 \vartheta \cdot \sin^2 \vartheta = 0 \text{ se } 2\alpha-1 > 0 \Rightarrow$$

$\Rightarrow \alpha > \frac{1}{2}$. La funzione è differentiabile in $(0; 0)$ solo se $\alpha > \frac{1}{2}$.

IM3) $f(x; y) = \alpha x^2 + \beta y^3 + 2x - 3y^2$.

$$f'_x = 2\alpha x + 2; f''_{xx} = 2\alpha; f'_y = 3\beta y^2 - 6y; f''_{yy} = 6\beta y - 6. \text{ Quindi:}$$

$$f'_x(1; 1) = 2\alpha + 2 = f''_{yy}(1; 1) = 6\beta - 6; f'_y(-1; -1) = 3\beta + 6 = f''_{xx}(-1; -1) = 2\alpha. \text{ Per cui:}$$

$$\begin{cases} 2\alpha + 2 = 6\beta - 6 \\ 3\beta + 6 = 2\alpha \end{cases} \Rightarrow \begin{cases} 2\alpha - 6\beta = -8 \\ 2\alpha - 3\beta = 6 \end{cases} \Rightarrow \begin{cases} -3\beta = -14 \\ 2\alpha = 3\beta + 6 \end{cases} \Rightarrow \begin{cases} \beta = \frac{14}{3} \\ \alpha = \frac{1}{2} (3 \cdot \frac{14}{3} + 6) = \frac{1}{2} \cdot 20 = 10. \end{cases}$$

IM4) $f(x; y) = y \cdot \log x + (x-1) e^y - y = 0. f(1; 0) = 0 + 0 - 0 = 0.$

AM2

$$\nabla f(x; y) = \left(y \cdot \frac{1}{x} + e^y; \log x + (x-1)e^y - 1 \right); \nabla f(1; 0) = (1; -1).$$

\exists possibile definire $x \rightarrow y(x)$ in quanto $f'_y(1; 0) = -1 \neq 0$, mentre $y'(1) = -\frac{1}{-1} = 1$.

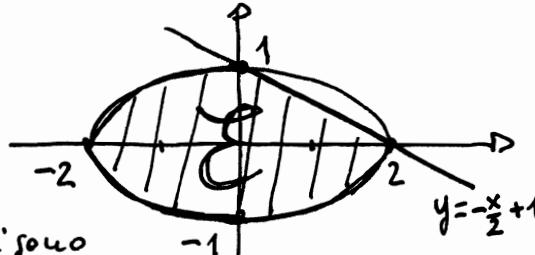
$$H(x; y) = \begin{vmatrix} -y \cdot \frac{1}{x^2} & \frac{1}{x} + e^y \\ \frac{1}{x} + e^y & (x-1) \cdot e^y \end{vmatrix}; H(1; 0) = \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}.$$

$$\text{Dove } y'' = -\frac{f''_{xx} + 2f''_{xy} \cdot y' + f''_{yy} \cdot (y')^2}{f'_y} \Rightarrow y''(1) = -\frac{0+2 \cdot 2 \cdot 1 + 0 \cdot 1}{-1} = 4.$$

$$P_2(x; 1) = 0 + y'(1)(x-1) + \frac{1}{2} y''(1)(x-1)^2 = x-1 + 2(x-1)^2 = 2x^2 - 3x + 1.$$

$$\text{II M1}) \begin{cases} \text{Max/min } f(x; y) = x^2 - y^2 \\ \text{s.r. } \begin{cases} x^2 + 4y^2 \leq 4 \\ 2y \leq 2-x \end{cases} \end{cases} \Rightarrow \begin{cases} \text{Max/min } f(x; y) = x^2 - y^2 \\ \text{s.v. } \begin{cases} x^2 + 4y^2 - 4 \leq 0 \\ x + 2y - 2 \leq 0 \end{cases} \end{cases}.$$

La funzione $f(x; y)$ è continua, Σ è compatto, i vincoli sono qualificate.



$$\Lambda(x; y; \lambda_1; \lambda_2) = x^2 - y^2 - \lambda_1(x^2 + 4y^2 - 4) - \lambda_2(x + 2y - 2).$$

$$\text{Caso } \lambda_1 = \lambda_2 = 0: \begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = -2y = 0 \\ x^2 + 4y^2 \leq 4 \\ 2y \leq 2-x \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ 0+0 \leq 4 \\ 0 \leq 2-0 \end{cases}; H(x; y) = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = H(0; 0); f''_{xx} \cdot f''_{yy} < 0: \text{P. Sella.}$$

$$\text{Caso } \lambda_1 \neq 0; \lambda_2 = 0: \begin{cases} \Lambda'_x = 2x - 2\lambda_1 x = 2x(1-\lambda_1) = 0 \\ \Lambda'_y = -2y - 8\lambda_1 y = -2y(1+4\lambda_1) = 0 \\ x^2 + 4y^2 \leq 4 \\ 2y \leq 2-x \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ 0+0=4 \\ 0 \leq 2-0 \end{cases} \cup \begin{cases} x=0 \\ \lambda_1 = -\frac{1}{4} \\ y = \pm 1 \\ \pm 2 \leq 2-0 \end{cases} \cup \begin{cases} \lambda_1 = 1 \\ y=0 \\ x = \pm 2 \\ 0 \leq 2 \mp 2 \end{cases}$$

$$\text{Caso } \lambda_1 = 0; \lambda_2 \neq 0: \begin{cases} \Lambda'_x = 2x - \lambda_2 = 0 \\ \Lambda'_y = -2y - 2\lambda_2 = 0 \\ 2y = 2-x \\ x^2 + 4y^2 \leq 4 \end{cases} \Rightarrow \begin{cases} x = \frac{\lambda_2}{2} \\ y = -\lambda_2 \\ -2\lambda_2 = 2 - \frac{\lambda_2}{2} \Rightarrow \lambda_2 = -\frac{4}{3} \\ x^2 + 4y^2 \leq 4 \end{cases} \Rightarrow \begin{cases} x = -\frac{2}{3} \\ y = +\frac{4}{3} \\ \lambda_2 = -\frac{4}{3} \\ \frac{4}{9} + \frac{64}{9} = \frac{68}{9} \leq 4 \text{ NO} \end{cases}$$

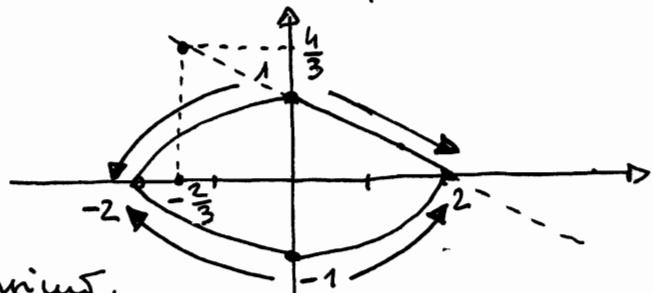
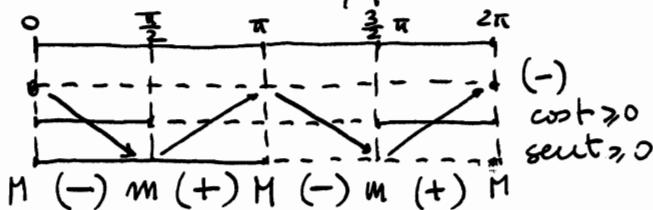
$$\text{Caso } \lambda_1 \neq 0; \lambda_2 \neq 0: \begin{cases} \Lambda'_x = 2x - 2\lambda_1 x - \lambda_2 = 0 \\ \Lambda'_y = -2y - 8\lambda_1 y - 2\lambda_2 = 0 \\ x^2 + 4y^2 = 4 \\ 2y = 2-x \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \cup \\ x=2 \\ y=0 \end{cases} \Rightarrow$$

AM3

$$\begin{cases} x=0 \\ y=1 \\ -\lambda_2=0 \\ -2-8\lambda_1-2\lambda_2=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \\ \lambda_1=-\frac{1}{2} \\ \lambda_2=0 \end{cases} \text{ Min?} \cup \begin{cases} x=2 \\ y=0 \\ 4-4\lambda_1-\lambda_2=0 \Rightarrow \lambda_1=1 \\ -2\lambda_2=0 \\ \lambda_2=0 \end{cases} \text{ Max?}$$

Studio sulla frontiera. Se $x=2-2y \Rightarrow f(y) = 4+4y^2-8y-y^2 = 3y^2-8y+4 \Rightarrow f'(y) = 6y-8 \geq 0$ per $y \geq \frac{4}{3}$. Sulle rette la funzione è sempre crescente.

Se $x^2+4y^2=4$ ponendo $\begin{cases} x=2\cos t \\ y=\sin t \end{cases} \Rightarrow f(t) = 4\cos^2 t - \sin^2 t = 5\cos^2 t - 1 \Rightarrow f'(t) = 10\cos t(-\sin t) > 0$.



Quindi i punti $(2; 0)$ e $(-2; 0)$ sono punti di massimo,

con $f(2; 0) = f(-2; 0) = 4$ mentre i punti $(0; 1)$ e $(0; -1)$ sono punti di minimo, con $f(0; 1) = f(0; -1) = -1$.

$$\text{II M2) } \begin{cases} y''' - y' = 3 \\ y(0) = 1 \\ y'(0) = 1 \\ y''(0) = 1 \end{cases} \quad \text{Equazione omogenea: } y''' - y' = 0 \Rightarrow \lambda^3 - \lambda = \lambda(\lambda^2 - 1) = 0 \Rightarrow \lambda_1 = 0; \lambda_2 = 1; \lambda_3 = -1. \text{ Soluzione generale dell'omogenea: } y(x) = c_1 + c_2 e^x + c_3 e^{-x}. \text{ Troviamo una soluzione particolare per la non omogenea. Avremo: } \mathcal{D}(\mathcal{D}^3 - \mathcal{D})(y) = \mathcal{D}(3) = 0 \text{ quindi occorre considerare l'annichilatrice } \mathcal{D}^2 \text{ ed avremo: } y_0(x) = ax + b \Rightarrow y_0' = a; y_0'' = y_0''' = 0.$$

Quindi: $y''' - y' = 3 \Rightarrow 0 - a = 3 \Rightarrow a = -3$. La soluzione generale sarà quindi:

$$y(x) = c_1 + c_2 e^x + c_3 e^{-x} - 3x. \text{ Da } y'(x) = c_2 e^x - c_3 e^{-x} + 3 \text{ e } y''(x) = c_2 e^x + c_3 e^{-x} \text{ si ha:}$$

$$\begin{cases} y(0) = c_1 + c_2 + c_3 = 1 \\ y'(0) = c_2 - c_3 - 3 = 1 \\ y''(0) = c_2 + c_3 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 - c_3 = 4 \\ c_2 + c_3 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 - 1 + c_2 = 4 \\ c_3 = 1 - c_2 \end{cases} \Rightarrow \begin{cases} c_1 = 0 \\ c_2 = \frac{5}{2} \\ c_3 = -\frac{3}{2} \end{cases} \text{ e quindi}$$

la soluzione del problema di Cauchy è $y(x) = \frac{5}{2}e^x - \frac{3}{2}e^{-x} - 3x$.

AM4

$$\text{II M3}) y' \cdot e^{x+y} = x^2 \Rightarrow y' \cdot e^x \cdot e^y = x^2 \Rightarrow y' \cdot e^y = x^2 \cdot e^{-x}.$$

$$\int e^y y' dx = \int e^y dy = \int x^2 e^{-x} dx + K = -x^2 e^{-x} + \int 2x e^{-x} dx + K \Rightarrow \\ \Rightarrow e^y = -x^2 e^{-x} - 2x e^{-x} + 2 \int e^{-x} dx + K = K - x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \Rightarrow \\ \Rightarrow y = \log(K - x^2 e^{-x} - 2x e^{-x} - 2e^{-x}) = \log(K - e^{-x}(x^2 + 2x + 2)).$$

$$\text{II M4}) \iint_D x^2 + y^2 dx dy ; D = \{(x, y) : x^2 - 1 \leq y \leq 1 - x^2\}$$

$$\iint_D x^2 + y^2 dx dy = \int_{-1}^{+1} \int_{x^2 - 1}^{1 - x^2} x^2 + y^2 dy dx = \\ = \int_{-1}^{+1} \left(x^2 y + \frac{1}{3} y^3 \Big|_{x^2 - 1}^{1 - x^2} \right) dx = \int_{-1}^{+1} \left[x^2(1 - x^2) + \frac{1}{3}(1 - x^2)^3 \right] - \left[x^2(x^2 - 1) + \frac{1}{3}(x^2 - 1)^3 \right] dx = \\ = \int_{-1}^{+1} 2x^2(1 - x^2) + \frac{2}{3}(1 - x^2)^3 dx = \int_{-1}^{+1} \cancel{2x^2} - \cancel{2x^4} + \frac{2}{3} - \cancel{2x^2} + \cancel{2x^4} - \frac{2}{3}x^6 dx = \\ = \frac{2}{3} \int_{-1}^{+1} 1 - x^6 dx = \frac{2}{3} \left(x - \frac{1}{7}x^7 \Big|_{-1}^1 \right) = \frac{2}{3} \left[\left(1 - \frac{1}{7} \right) - \left(-1 + \frac{1}{7} \right) \right] = \frac{2}{3} \cdot \left(2 - \frac{2}{7} \right) = \frac{8}{7}.$$

