TASK MATHEMATICS for ECONOMIC APPLICATIONS
11/11/2017

I M 1) $z = \dfrac{9}{i-1} + \dfrac{9}{1+i} = \dfrac{9(1+i)+9(i-1)}{(i-1)(1+i)} = \dfrac{18i}{i^2-1} = \dfrac{18i}{-2} = -9i$. The number z in trigonometric form is $z = 9\left(\cos\left(\dfrac{3}{2}\pi\right) + i\,\mathrm{sen}\left(\dfrac{3}{2}\pi\right)\right)$ and the three cubic roots of z are $\sqrt[3]{z} = \sqrt[3]{-9i} = \sqrt[3]{9\left(\cos\left(\dfrac{3}{2}\pi\right) + i\,\mathrm{sen}\left(\dfrac{3}{2}\pi\right)\right)} =$

$$= \sqrt[3]{9}\left(\cos\left(\frac{3\pi/2+2k\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{3\pi/2+2k\pi}{3}\right)\right) \text{ with } k = 0,1,2.$$

The three roots are: $z_1 = \sqrt[3]{9}\left(\cos\dfrac{\pi}{2} + i\,\mathrm{sen}\dfrac{\pi}{2}\right) = \sqrt[3]{9}\,i$;

$$z_2 = \sqrt[3]{9}\left(\cos\frac{7}{6}\pi + i\,\mathrm{sen}\frac{7}{6}\pi\right) = \sqrt[3]{9}\left(-\frac{\sqrt{3}}{2} - i\,\frac{1}{2}\right) = -\frac{\sqrt[3]{9}}{2}\left(\sqrt{3}+i\right);$$

$$z_3 = \sqrt[3]{9}\left(\cos\frac{11}{6}\pi + i\,\mathrm{sen}\frac{11}{6}\pi\right) = \sqrt[3]{9}\left(\frac{\sqrt{3}}{2} - i\,\frac{1}{2}\right) = \frac{\sqrt[3]{9}}{2}\left(\sqrt{3}-i\right).$$

I M 2) If the vector $(1,1,2,-1)$ belongs to the Kernel of F, matrix $\mathbb{A}$ must satisfy the condition $\begin{Vmatrix} 1 & 2 & -1 & 1 \\ 2 & 2 & 1 & k \\ 1 & 2 & m & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \\ 2 \\ -1 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix}$ or $\begin{Vmatrix} 0 \\ 6-k \\ 2+2m \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix}$ and it easily follows $k = 6$ and $m = -1$.

For such values we get $F(1,1,2,2) = \begin{Vmatrix} 1 & 2 & -1 & 1 \\ 2 & 2 & 1 & 6 \\ 1 & 2 & -1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \\ 2 \\ 2 \end{Vmatrix} = \begin{Vmatrix} 3 \\ 18 \\ 3 \end{Vmatrix}$.

The dimensions of the Image and the Kernel are both equal to 2, since in the matrix $\mathbb{A}$ the first and the third rows are equal while the second row is not proportional to the first and so $\mathrm{Dim}\,(\mathrm{Imm}\,(F)) = \mathrm{Rank}\,(\mathbb{A}) = 2 \Rightarrow \mathrm{Dim}\,(\mathrm{Ker}) = 4-2 = 2$.

I M 3) If $1+i$ is an eigenvalue of the matrix $\mathbb{A}$, the determinant $|\mathbb{A}-(1+i)\mathbb{I}|$ must be equal to 0: $|\mathbb{A}-(1+i)\mathbb{I}| = 0$.

$$\begin{vmatrix} 3-(1+i) & 1 & 1 \\ 2 & 4-(1+i) & 2 \\ 1 & m & -3-(1+i) \end{vmatrix} = \begin{vmatrix} 2-i & 1 & 1 \\ 2 & 3-i & 2 \\ 1 & m & -4-i \end{vmatrix} =$$

$$= (2-i)\begin{vmatrix} 3-i & 2 \\ m & -4-i \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 1 & -4-i \end{vmatrix} + \begin{vmatrix} 2 & 3-i \\ 1 & m \end{vmatrix} =$$

$$= (2-i)((3-i)(-4-i)-2m) - (2(-4-i)-2) + (2m-(3-i)) =$$

$$= (2-i)\left(i^2+i-12-2m\right) - (-8-2i-2) + 2m-3+i =$$

$$= (2-i)(i-13-2m) - (-10-2i) + 2m-3+ +i =$$

$$= -i^2+15i+2mi-26-4m+10+2i+2m-3+i =$$

$$= 18i-18+2mi-2m = 2(9(i-1)+m(i-1)) = 2(i-1)(9+m).$$

The determinant is zero if and only if $m = -9$.

To find the last eigenvalue of the matrix ($1-i$ is clearly the second eigenvalue) we calculate the characteristic polynomial of $\mathbb{A}$:

$$p_{\mathbb{A}}(\lambda)=|\mathbb{A}-\lambda\mathbb{I}|=\begin{vmatrix}3-\lambda & 1 & 1\\ 2 & 4-\lambda & 2\\ 1 & -9 & -3-\lambda\end{vmatrix}=$$

$$=(3-\lambda)\begin{vmatrix}4-\lambda & 2\\ -9 & -3-\lambda\end{vmatrix}-\begin{vmatrix}2 & 2\\ 1 & -3-\lambda\end{vmatrix}+\begin{vmatrix}2 & 4-\lambda\\ 1 & -9\end{vmatrix}=$$

$$=(3-\lambda)\left(\lambda^2-\lambda+6\right)-(-2\lambda-8)+(\lambda-22)=$$

$$=-\lambda^3+4\lambda^2-9\lambda+18+2\lambda+8+\lambda-22=-\lambda^3+4\lambda^2-6\lambda+4$$

that can be easily factorized as $(2-\lambda)\left(\lambda^2-2\lambda+2\right)$.

The three eigenvalues of $\mathbb{A}$ are $\lambda_1=2$ and $\lambda_{2,3}=1\pm i$.

I M 4) If the vector $\mathbb{X}$ has coordinates $(2,1,2)$ in the basis $\{\mathbb{W}_1;\mathbb{W}_2;\mathbb{W}_3\}$ it results: $2\mathbb{W}_1+\mathbb{W}_2+2\mathbb{W}_3=\mathbb{X}$ from which we get $\mathbb{W}_3=\frac{1}{2}(\mathbb{X}-2\mathbb{W}_1-\mathbb{W}_2)=$

$$=\frac{1}{2}\left(\left\|\begin{matrix}1\\1\\1\end{matrix}\right\|-2\left\|\begin{matrix}1\\1\\-1\end{matrix}\right\|-\left\|\begin{matrix}1\\0\\1\end{matrix}\right\|\right)=\left\|\begin{matrix}-1\\-1/2\\1\end{matrix}\right\|.$$

II M 1) $\nabla f=\left(2x-y^2-1,2y-2xy\right)$.

I OC: $\begin{cases}2x-y^2-1=0\\2y-2xy=0\end{cases}\Rightarrow\begin{cases}2x-y^2-1=0\\2y(1-x)=0\end{cases}\Rightarrow\begin{cases}x=1/2\\y=0\end{cases}\cup\begin{cases}y^2=1\\x=1\end{cases}\Rightarrow$

$\begin{cases}x=1/2\\y=0\end{cases}\cup\begin{cases}y=\pm1\\x=1\end{cases}$.

We get three stationary points: $P_1=(1/2,0)$ and $P_{2,3}=(1,\pm1)$.

$\mathbb{H}(f)=\left\|\begin{matrix}2 & -2y\\ -2y & 2-2x\end{matrix}\right\|$; $|\mathbb{H}(f)|=4-4x-4y^2$.

II OC: $\begin{cases}f''_{x,x}(P_1)=2>0\\|\mathbb{H}f(P_1)|=2>0\end{cases}\Rightarrow P_1$ is a minimum point.

$|\mathbb{H}f(P_{2,3})|=-4<0\Rightarrow P_{2,3}$ are saddle points.

II M 2) $f(x,y)=xy-e^{y-x}=0$. Since $f(1,1)=0$, the equation is satisfied at point $(1,1)$. $\nabla f=(y+e^{y-x},x-e^{y-x})$ and $\nabla f(1,1)=(2,0)$. Only $f'_x(1,1)\neq0$; thus the equation $f(x,y)=0$ defines in a neighborhood of $y=1$ an implicit function $x=x(y)$ with $x'(1)=-\dfrac{f'_y(1,1)}{f'_x(1,1)}=-\dfrac{0}{2}=0$. For the second order derivative, since $\mathbb{H}(x,y)=\left\|\begin{matrix}-e^{y-x} & 1+e^{y-x}\\ 1+e^{y-x} & -e^{y-x}\end{matrix}\right\|$ and $\mathbb{H}(1,1)=\left\|\begin{matrix}-1 & 2\\ 2 & -1\end{matrix}\right\|$, we get:

$$x''(1)=-\frac{f''_{xx}(x')^2+2f''_{xy}x'+f''_{yy}}{f'_y}=-\frac{-1\cdot0+2\cdot2\cdot0+(-1)}{f'_y}=+\frac{1}{2}.$$

Since $x'(1)==0$ and $x''(1)=+\frac{1}{2}$ the point is a minimum point.

II M 3) $\nabla f=(e^x+1,-e^y-1)$, $\nabla f(0,0)=(2,-2)$ and so:

$D_vf(0,0)=\nabla f(0,0)\cdot v=(2,-2)\cdot(\cos\alpha,\operatorname{sen}\alpha)=2(\cos\alpha-\operatorname{sen}\alpha)$.

$D_vf(0,0)=0$ if and only if $\cos\alpha=\operatorname{sen}\alpha$ and this is true for $\alpha=\frac{\pi}{4}$ or $\alpha=\frac{5\pi}{4}$.

For $D^2_{v,v}f(0,0)$ we calculate $\mathbb{H}(f) = \left\|\begin{matrix} e^x & 0 \\ 0 & -e^y \end{matrix}\right\| \Rightarrow \mathbb{H}f(0,0) = \left\|\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}\right\|$. So

$$D^2_{v,v}f(0,0) = v \cdot \mathbb{H}f(0,0) \cdot v^T = (\cos\alpha, \text{sen}\,\alpha) \cdot \left\|\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}\right\| \cdot \begin{pmatrix} \cos\alpha \\ \text{sen}\,\alpha \end{pmatrix} =$$

$$= \cos^2\alpha - \text{sen}^2\alpha = (\cos\alpha - \text{sen}\,\alpha)(\cos\alpha + \text{sen}\,\alpha) = 0 \text{ since } \cos\alpha = \text{sen}\,\alpha.$$

II M 4) $\begin{cases} \text{Max/min } f(x,y) = x^2 + y^2 \\ \text{u.c. } 0 \le y \le 1 - x^2 \end{cases}$. The objective function of the problem is increasing on both variables from the origin point $(0,0)$ that belongs to the feasible region of the problem (red-drawed in the figure in the next page).

Easily we see that $min(f) = f(0,0) = 0$. For $Max(f)$ again by increasing on both variables from the origin point $(0,0)$ we note that $Max(f)$ must be found on the upper boundary of the admissible region, thus the problem: $\begin{cases} \text{Max } f(x,y) = x^2 + y^2 \\ \text{u.c. } 0 \le y \le 1 - x^2 \end{cases}$ is equivalent to $\begin{cases} \text{Max } f(x, 1-x^2) = x^2 + (1-x^2)^2 \\ \text{u.c. } -1 \le x \le 1 \end{cases}$ and by simmetry the last problem is equivalent to $\begin{cases} \text{Max } g(x) = x^4 - x^2 + 1 \\ \text{u.c. } 0 \le x \le 1 \end{cases}$.

Since $g'(x) = 4x^3 - 2x = 2x(2x^2 - 1)$, for $0 \le x \le 1$, $g'(x) \ge 0$ if and only if $2x^2 - 1 \ge 0 \Leftrightarrow x^2 \ge 1/2 \Leftrightarrow x \ge \sqrt{1/2}$.

The function $g(x)$ is increasing in $[\sqrt{1/2}, 1]$ while is decreasing in $[0, \sqrt{1/2}]$.

We conclude that $Max(f) = Max(g) = Max\{g(0), g(1)\} = 1$ at points $(0,1)$ or $(\pm 1, 0)$.

On the figure below are drawed the positive level curves (blue).