TASK MATHEMATICS for ECONOMIC APPLICATIONS

15/1/2018

I M 1) $z = e^{log\,16 - i\pi} = e^{log\,16}\,e^{-i\pi} = 16\,(\cos(-\pi) + i\sin(-\pi)) =$
$= 16\,(\cos\pi + i\sin\pi) = -16$. So $\sqrt[4]{z} = \sqrt[4]{-16} = \sqrt[4]{16}\sqrt[4]{\cos\pi + i\sin\pi} =$
$= 2\left(\cos\left(\frac{\pi}{4} + k\,\frac{2\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + k\,\frac{2\pi}{4}\right)\right)$ with $k = 0, 1, 2, 3$.
The four roots are:
for $k = 0: 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2} + i\sqrt{2}$;
for $k = 1: 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\sqrt{2} + i\sqrt{2}$;
for $k = 2: 2\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\sqrt{2} - i\sqrt{2}$;
for $k = 3: 2\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt{2} - i\sqrt{2}$.

I M 2) To find the eigenvalues of the matrix $\mathbb{A}$, we need determinant $|\mathbb{A} - \lambda\,\mathbb{I}| = 0$:

$$\begin{vmatrix} -\lambda & 0 & k \\ 0 & k-\lambda & 0 \\ k & 0 & -\lambda \end{vmatrix} = (k-\lambda)\left(\lambda^2 - k^2\right) = (k-\lambda)(\lambda-k)(\lambda+k) = 0.$$

So the three eigenvalues are $\lambda_1 = \lambda_2 = k, \lambda_3 = -k$.

For $k \neq 0$ it is $m^a_k = 2,\ m^a_{-k} = 1$. Since $\|\mathbb{A} - k\,\mathbb{I}\| = \begin{Vmatrix} -k & 0 & k \\ 0 & 0 & 0 \\ k & 0 & -k \end{Vmatrix}$ we get
$m^g_k = 3 - \text{Rank}\,(\|\mathbb{A} - k\,\mathbb{I}\|) = 3 - 1 = 2$ (obviously, since the matrix is symmetric).
Since $\lambda_3 = -k$ is a simple eigenvalue, $m^g_{-k} = m^a_{-k} = 1$.
If $k = 0$ we easily get $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $m^g_0 = m^a_0 = 3$.
For $k \neq 0$, to find eigenvectors corresponding to $\lambda_1 = \lambda_2 = k$ we solve the system:

$$\|\mathbb{A} - k\,\mathbb{I}\| \cdot \mathbb{X} = \begin{Vmatrix} -k & 0 & k \\ 0 & 0 & 0 \\ k & 0 & -k \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x - z = 0 \\ \forall\, y \end{cases}$$

so eigenvectors are: $\mathbb{V} = (x, y, x)$.
For the basis we need two linearly independent eigenvectors, and we choose:
$\mathbb{V}_1 = (1, 0, 1)$ and $\mathbb{V}_2 = (0, 1, 0)$.
For $k \neq 0$, to find eigenvectors corresponding to $\lambda_3 = -k$ we solve the system:

$$\|\mathbb{A} + k\,\mathbb{I}\| \cdot \mathbb{X} = \begin{Vmatrix} k & 0 & k \\ 0 & 2k & 0 \\ k & 0 & k \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x + z = 0 \\ y = 0 \end{cases}$$

so the unique eigenvector is $\mathbb{V} = (x, 0, -x)$.
For the basis we choose $\mathbb{V}_3 = (1, 0, -1)$.

If $k = 0 \Rightarrow \mathbb{A} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix}$, the null matrix, and its eigenvectors are all $\mathbb{V} \in \mathbb{R}^3$.

I M 3) Since $\begin{cases} f(1,1,1)=(3,3,1) \\ f(1,1,-1)=(-1,5,-1) \end{cases}$ we get:

$$\left\|\begin{matrix} 1 & x_1 & y_1 \\ 2 & x_2 & y_2 \\ -1 & x_3 & y_3 \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}\right\| = \left\|\begin{matrix} 3 \\ 3 \\ 1 \end{matrix}\right\| \Rightarrow \begin{cases} 1+x_1+y_1=3 \\ 2+x_2+y_2=3 \\ -1+x_3+y_3=1 \end{cases} \Rightarrow \begin{cases} x_1+y_1=2 \\ x_2+y_2=1 \\ x_3+y_3=2 \end{cases} \text{and}$$

$$\left\|\begin{matrix} 1 & x_1 & y_1 \\ 2 & x_2 & y_2 \\ -1 & x_3 & y_3 \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 \\ 1 \\ -1 \end{matrix}\right\| = \left\|\begin{matrix} -1 \\ 5 \\ -1 \end{matrix}\right\| \Rightarrow \begin{cases} 1+x_1-y_1=-1 \\ 2+x_2-y_2=5 \\ -1+x_3-y_3=-1 \end{cases} \Rightarrow \begin{cases} x_1-y_1=-2 \\ x_2-y_2=3 \\ x_3-y_3=0 \end{cases}$$

and so:

$$\begin{cases} x_1+y_1=2 \\ x_1-y_1=-2 \end{cases} \Rightarrow \begin{cases} x_1=0 \\ y_1=2 \end{cases};\ \begin{cases} x_2+y_2=1 \\ x_2-y_2=3 \end{cases} \Rightarrow \begin{cases} x_2=2 \\ y_2=-1 \end{cases};$$

$\begin{cases} x_3+y_3=2 \\ x_3-y_3=0 \end{cases} \Rightarrow \begin{cases} x_3=1 \\ y_3=1 \end{cases}$. So $\mathbb{A} = \left\|\begin{matrix} 1 & 0 & 2 \\ 2 & 2 & -1 \\ -1 & 1 & 1 \end{matrix}\right\|$. Using elementary operations on the rows: $R_2 \leftarrow R_2 - 2R_1$; $R_3 \leftarrow R_3 + 2R_1$ and then $R_3 \leftarrow R_3 - \frac{1}{2}R_2$ we get:

$$\left\|\begin{matrix} 1 & 0 & 2 \\ 2 & 2 & -1 \\ -1 & 1 & 1 \end{matrix}\right\| \Rightarrow \left\|\begin{matrix} 1 & 0 & 2 \\ 0 & 2 & -5 \\ 0 & 1 & 3 \end{matrix}\right\| \Rightarrow \left\|\begin{matrix} 1 & 0 & 2 \\ 0 & 2 & -5 \\ 0 & 0 & \frac{11}{2} \end{matrix}\right\|$$

. So $|\mathbb{A}| \neq 0$ and $\text{Rank}(\mathbb{A}) = 3$.

From this we get $\text{Dim}(\text{Imm}(\mathbb{A})) = 3$ and $\text{Dim}(\text{Ker}(\mathbb{A})) = 3 - 3 = 0$.

I M 4) If the vector $\mathbb{X}$ has coordinates $(2,-1,2)$ in the basis $\mathbb{W}$ it means that:

$$\mathbb{X} = \left\|\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} 2 \\ -1 \\ 1 \end{matrix}\right\| = \left\|\begin{matrix} 2 \\ 0 \\ 3 \end{matrix}\right\|$$

. If we use the basis $\mathbb{V}$ we get:

$$\mathbb{X} = \left\|\begin{matrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{matrix}\right\| \cdot \left\|\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\| = \left\|\begin{matrix} 2 \\ 0 \\ 3 \end{matrix}\right\| \Rightarrow \begin{cases} x_1+x_2+2x_3=2 \\ 0\,x_1+x_2+x_3=0 \\ x_1+0\,x_2+2x_3=3 \end{cases} \Rightarrow \begin{cases} x_1=1 \\ x_2=-1 \\ x_3=1 \end{cases}.$$

In fact $\left\|\begin{matrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{matrix}\right\| \cdot \left\|\begin{matrix} 1 \\ -1 \\ 1 \end{matrix}\right\| = \left\|\begin{matrix} 2 \\ 0 \\ 3 \end{matrix}\right\| = \mathbb{X}$.

II M 1) From the equation $f(x,y) = y\log x + (x-1)\,e^y - y = 0$ we get $f(1,0) = 0$.

$\nabla f(x,y) = \left(y \cdot \frac{1}{x} + e^y, \log x + (x-1)\,e^y - 1\right) \Rightarrow \nabla f(1,0) = (1,-1)$.

Since $f'_y(1,0) = -1 \neq 0$, we can define an implicit function $y = y(x)$, for which:

$y'(1) = -\frac{f'_x(1,0)}{f'_y(1,0)} = -\frac{1}{-1} = 1$.

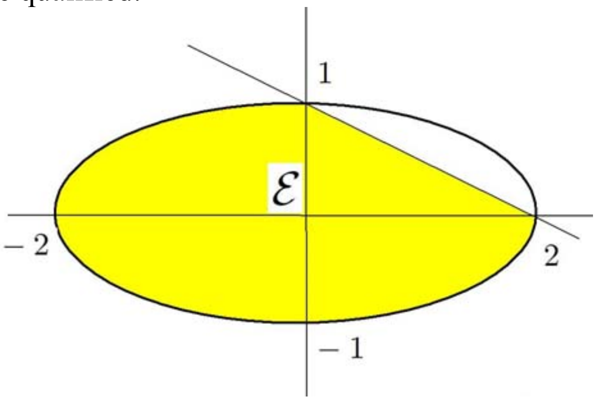
Furthermore $\mathbb{H}(x,y) = \left\|\begin{matrix} -y \cdot \frac{1}{x^2} & \frac{1}{x} + e^y \\ \frac{1}{x} + e^y & (x-1)\,e^y \end{matrix}\right\| \Rightarrow \mathbb{H}(1,0) = \left\|\begin{matrix} 0 & 2 \\ 2 & 0 \end{matrix}\right\|$ and so:

$y''(1) = -\frac{f''_{xx} + 2f''_{xy}\,y' + f''_{yy}\,(y')^2}{f'_y} = -\frac{0 + 2\cdot 2\cdot 1 + 0\cdot 1}{-1} = 4$. For the expression of the Taylor polynomial of the second degree at $x = 1$ we get:

$P_2(x,1) = 0 + y'(1)(x-1) + \frac{1}{2}y''(1)(x-1)^2 = x - 1 + 2(x-1)^2 = 2x^2 - 3x + 1$.

II M 2) Firstly we write the problem $\begin{cases} \text{Max/min } f(x,y) = x^2 - y^2 \\ \text{u.c.} \begin{cases} x^2 + 4y^2 \le 4 \\ 2y \le 2 - x \end{cases} \end{cases}$ in the form:

$\begin{cases} \text{Max/min } f(x,y) = x^2 - y^2 \\ \text{u.c.} \begin{cases} x^2 + 4y^2 - 4 \le 0 \\ x + 2y - 2 \le 0 \end{cases} \end{cases}$. The objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a compact set, and so maximum and minimum values surely exist. The constraints are qualified.



The Lagrangian function of the problem is:

$\Lambda(x,y,\lambda_1,\lambda_2) = x^2 - y^2 - \lambda_1(x^2 + 4y^2 - 4) - \lambda_2(x + 2y - 2)$.

1) case $\lambda_1 = 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = -2y = 0 \\ x^2 + 4y^2 \le 4 \\ 2y \le 2 - x \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 + 0 \le 4 \\ 0 \le 2 - 0 \end{cases}. \quad \mathbb{H}(x,y) = \mathbb{H}(0,0) = \left\| \begin{matrix} 2 & 0 \\ 0 & -2 \end{matrix} \right\|.$$

Since $f''_{xx} \cdot f''_{yy} < 0$ the point $(0,0)$ is a saddle point.

2) case $\lambda_1 \ne 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_1 x = 2x(1 - \lambda_1) = 0 \\ \Lambda'_y = -2y - 8\lambda_1 y = -2y(1 + 4\lambda_1) = 0 \\ x^2 + 4y^2 = 4 \\ 2y \le 2 - x \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 + 0 = 4 \\ 0 \le 2 - 0 \end{cases} \text{impossible}$$

$$\cup \begin{cases} x = 0 \\ \lambda_1 = -\frac{1}{4} \\ 4y^2 = 4 \\ 2y \le 2 - x \end{cases} \Rightarrow \begin{cases} x = 0 \\ \lambda_1 = -\frac{1}{4} < 0 \\ y = \pm 1 \\ \pm 2 \le 2 - 0 \end{cases} : \ (0,1) \text{ and } (0,-1) \text{ may be minimum points;}$$

$$\cup \begin{cases} \lambda_1 = 1 \\ y = 0 \\ x^2 = 4 \\ 2y \le 2 - x \end{cases} \Rightarrow \begin{cases} x = \pm 2 \\ y = 0 \\ \lambda_1 = 1 > 0 \\ 0 \le 2 \pm 2 \end{cases} : (2,0) \text{ and } (-2,0) \text{ may be maximum points;}$$

$$\cup \begin{cases} \lambda_1 = 1 \\ \lambda_1 = -\frac{1}{4} \\ x^2 + 4y^2 = 4 \\ 2y \le 2 - x \end{cases} \text{impossible}.$$

3) case $\lambda_1 = 0, \lambda_2 \ne 0$:

$$\begin{cases} \Lambda'_x = 2x - \lambda_2 = 0 \\ \Lambda'_y = -2y - 2\lambda_2 = 0 \\ 2y = 2 - x \\ x^2 + 4y^2 \le 4 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}\lambda_2 \\ y = -\lambda_2 \\ -2\lambda_2 + \frac{1}{2}\lambda_2 = 2 \\ x^2 + 4y^2 \le 4 \end{cases} \Rightarrow \begin{cases} x = -\frac{2}{3} \\ y = \frac{4}{3} \\ \lambda_2 = -\frac{4}{3} \\ \frac{4}{9} + \frac{64}{9} \le 4 : \text{not satisfied} \end{cases}.$$

The point $\left(-\dfrac{2}{3}, \dfrac{4}{3}\right) \notin \mathcal{E}$.

4) case $\lambda_1 \neq 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_1 x - \lambda_2 = 0 \\ \Lambda'_y = -2y - 8\lambda_1 y - 2\lambda_2 = 0 \\ x^2 + 4y^2 = 4 \\ x + 2y = 2 \end{cases}. \text{ From } \begin{cases} x^2 + 4y^2 = 4 \\ x + 2y = 2 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \end{cases} \cup \begin{cases} x = 2 \\ y = 0 \end{cases} \Rightarrow$$

$$\begin{cases} x = 0 \\ y = 1 \\ -\lambda_2 = 0 \\ -2y - 8\lambda_1 - 2\lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = -\frac{1}{4} \\ \lambda_2 = 0 \end{cases} \text{ just seen, it may be a minimum point;}$$

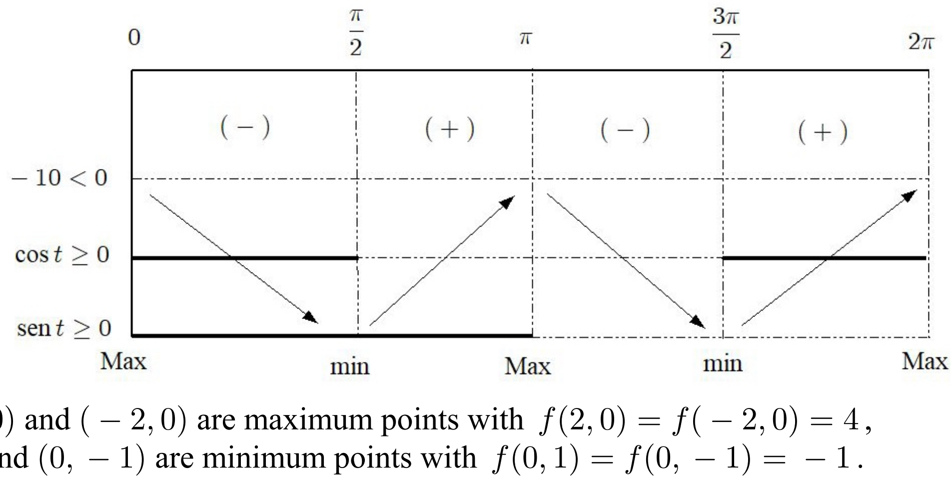
$$\begin{cases} x = 2 \\ y = 0 \\ 4 - 4\lambda_1 - \lambda_2 = 0 \\ -2\lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 0 \\ \lambda_1 = 1 \\ \lambda_2 = 0 \end{cases} \text{ just seen, it may be a maximum point.}$$

Now we study the function in the boundary points.

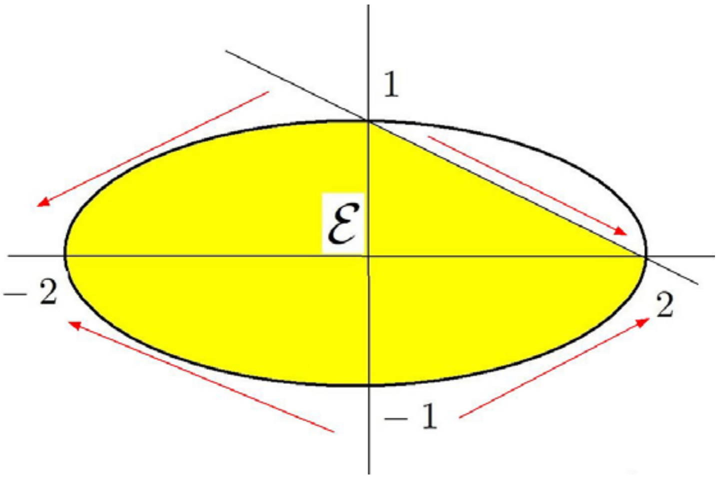
For $x + 2y = 2 \Rightarrow x = 2 - 2y$ we get $f(y) = (2 - 2y)^2 - y^2 = 3y^2 - 8y + 4$ from which $f'(y) = 6y - 8 \ge 0$ for $y \ge \dfrac{4}{3}$. So for $-1 \le y \le 1$ the function is increasing.

For $x^2 + 4y^2 = 4$ we put $\begin{cases} x = 2\cos t \\ y = \operatorname{sen} t \end{cases}$ to get $f(t) = 4\cos^2 t - \operatorname{sen}^2 t = 5\cos^2 t - 1$ from which $f'(t) = -10\cos t \operatorname{sen} t \ge 0$.

Since $\begin{cases} -10 < 0 \\ \cos t \ge 0 \text{ for } 0 \le t \le \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \le t \le 2\pi \\ \operatorname{sen} t \ge 0 \text{ for } 0 \le t \le \pi \end{cases}$ we get:



So $(2, 0)$ and $(-2, 0)$ are maximum points with $f(2, 0) = f(-2, 0) = 4$,
$(0, 1)$ and $(0, -1)$ are minimum points with $f(0, 1) = f(0, -1) = -1$.



II M 3) For the function $f(x,y) = y^2 - x^3 - 2xy^2 + 27x$ firstly we find its stationary points. $\nabla f = \left(-3x^2 - 2y^2 + 27, 2y - 4xy\right)$.

I OC: $\begin{cases} -3x^2 - 2y^2 + 27 = 0 \\ 2y - 4xy = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 + 2y^2 - 27 = 0 \\ 2y(1-2x) = 0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x^2 = 9 \\ y = 0 \end{cases} \Rightarrow \begin{cases} x = 3 \\ y = 0 \end{cases}$ and $\begin{cases} x = -3 \\ y = 0 \end{cases} \cup \begin{cases} y^2 = \frac{105}{8} \\ x = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ y = \frac{\sqrt{105}}{2\sqrt{2}} \end{cases}$ and $\begin{cases} x = \frac{1}{2} \\ y = -\frac{\sqrt{105}}{2\sqrt{2}} \end{cases}$

There are four stationary points: $(1,0)$, $(-1,0)$, $\left(\frac{1}{2}, \frac{\sqrt{105}}{2\sqrt{2}}\right)$, $\left(\frac{1}{2}, -\frac{\sqrt{105}}{2\sqrt{2}}\right)$.

II OC: $\mathbb{H}(x,y) = \left\| \begin{matrix} -6x & -4y \\ -4y & 2-4x \end{matrix} \right\|$.

$\mathbb{H}(3,0) = \left\| \begin{matrix} -18 & 0 \\ 0 & -10 \end{matrix} \right\| : \begin{cases} |\mathbb{H}_1| < 0 \\ |\mathbb{H}_2| > 0 \end{cases} :$ $(3,0)$ is a Maximum point;

$\mathbb{H}(-3,0) = \left\| \begin{matrix} 18 & 0 \\ 0 & 14 \end{matrix} \right\| : \begin{cases} |\mathbb{H}_1| > 0 \\ |\mathbb{H}_2| > 0 \end{cases} :$ $(-3,0)$ is a minimum point;

$\mathbb{H}\left(\frac{1}{2}, \frac{\sqrt{105}}{2\sqrt{2}}\right) = \left\| \begin{matrix} -3 & -\sqrt{210} \\ -\sqrt{210} & 0 \end{matrix} \right\| : |\mathbb{H}_2| < 0 : \left(\frac{1}{2}, \frac{\sqrt{105}}{2\sqrt{2}}\right)$ is a Saddle point.

$\mathbb{H}\left(\frac{1}{2}, -\frac{\sqrt{105}}{2\sqrt{2}}\right) = \left\| \begin{matrix} -3 & \sqrt{210} \\ \sqrt{210} & 0 \end{matrix} \right\| : |\mathbb{H}_2| < 0 : \left(\frac{1}{2}, -\frac{\sqrt{105}}{2\sqrt{2}}\right)$ is a Saddle point.

II M 4) From $f(x,y) = \alpha x^2 + \beta y^3 + 2x - 3y^2$ we get:

$f'_x(x,y) = 2\alpha x + 2$, $f''_{xx}(x,y) = 2\alpha$, $f'_y(x,y) = 3\beta y^2 - 6y$, $f''_{yy}(x,y) = 6\beta y - 6$.

From the given conditions:

$f'_x(1,1) = 2\alpha + 2 = f''_{yy}(1,1) = 6\beta - 6 \Rightarrow 2\alpha + 2 = 6\beta - 6 \Rightarrow 2\alpha - 6\beta = -8$;

$f'_y(-1,-1) = 3\beta + 6 = f''_{xx}(-1,-1) = 2\alpha \Rightarrow 3\beta + 6 = 2\alpha \Rightarrow 2\alpha - 3\beta = 6$.

$\begin{cases} 2\alpha - 6\beta = -8 \\ 2\alpha - 3\beta = 6 \end{cases} \Rightarrow \begin{cases} -3\beta = -14 \\ 2\alpha = 3\beta + 6 \end{cases} \Rightarrow \begin{cases} \beta = \frac{14}{3} \\ \alpha = 10 \end{cases}$.