TASK MATHEMATICS for ECONOMIC APPLICATIONS
12/2/2018

I M 1) $e^{z} = 1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right) = \sqrt{2}\Big(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\Big) \Rightarrow$

$\Rightarrow e^{z} = e^{x+iy} = e^{x}\,e^{iy} = \sqrt{2}\Big(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\Big) \Rightarrow \begin{cases} e^{x} = \sqrt{2} \Rightarrow x = \log\sqrt{2} \\ y = \frac{\pi}{4} \end{cases}$.

So $z = \log\sqrt{2} + i\,\frac{\pi}{4}$.

I M 2) To find the eigenvalues of the matrix $\mathbb{A}$, we need the determinant $|\mathbb{A} - \lambda\,\mathbb{I}| = 0$:

$$\begin{vmatrix} 3-\lambda & 2 & 1 \\ 1 & 4-\lambda & -9 \\ 1 & 2 & -3-\lambda \end{vmatrix} \overset{R_1 - R_3}{=} \begin{vmatrix} 2-\lambda & 0 & 4+\lambda \\ 1 & 4-\lambda & -9 \\ 1 & 2 & -3-\lambda \end{vmatrix} =$$

$= (2-\lambda)[(4-\lambda)(-3-\lambda)+18] + (4+\lambda)(2-4+\lambda) =$

$= (2-\lambda)\big(\lambda^2 - \lambda - 12 + 18 - 4 - \lambda\big) = (2-\lambda)\big(\lambda^2 - 2\lambda + 2\big) = 0$.

There is a real eigenvalue: $\lambda_1 = 2$. From $\big(\lambda^2 - 2\lambda + 2\big) = 0$ we get:

$\lambda = 1 \pm \sqrt{1-2} = 1 \pm \sqrt{-1} = 1 \pm i$. So there are two complex eigenvalues:

$\lambda_2 = 1+i$ and $\lambda_3 = 1-i$. Three simple (distinct) eigenvalues, one real and two complex.

To find eigenvectors corresponding to $\lambda_1 = 2$ we solve the system:

$$\|\mathbb{A} - 2\,\mathbb{I}\| \cdot \mathbb{X} = \begin{Vmatrix} 1 & 2 & 1 \\ 1 & 2 & -9 \\ 1 & 2 & -5 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{Vmatrix} 1 & 2 & 1 \\ 0 & 0 & -10 \\ 0 & 0 & -6 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow$$

$\Rightarrow \begin{cases} x + 2y + z = 0 \\ -10z = 0 \end{cases} \Rightarrow \begin{cases} x = -2y \\ z = 0 \end{cases}$ so the eigenvectors are: $\mathbb{V} = (2k, -k, 0)$.

I M 3) From $\begin{cases} x_1 + mx_2 + mx_3 + 2x_4 = 0 \\ x_1 + x_2 + x_3 + 2mx_4 = 0 \end{cases}$ we get:

$\begin{Vmatrix} 1 & m & m & 2 \\ 1 & 1 & 1 & 2m \end{Vmatrix} \overset{R_2 - R_1}{\Rightarrow} \begin{Vmatrix} 1 & m & m & 2 \\ 0 & 1-m & 1-m & 2m-2 \end{Vmatrix}$ and so:

for $m = 1 \Rightarrow \text{Rank}(\mathbb{A}) = 1 \Rightarrow$ the system has $\infty^{4-1} = \infty^3$ solutions ;

for $m \neq 1 \Rightarrow \text{Rank}(\mathbb{A}) = 2 \Rightarrow$ the system has $\infty^{4-2} = \infty^2$ solutions .

For $m = 1$ we get $\begin{Vmatrix} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \\ w \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \Rightarrow x + y + z + 2w = 0 \Rightarrow$

$\Rightarrow z = -x - y - 2w$. So a basis for the vector space of the solutions may be:

$\mathbb{V} = \{\mathbb{V}_1 = (1,0,-1,0); \mathbb{V}_2 = (0,1,-1,0); \mathbb{V}_3 = (0,0,-2,1)\}$.

I M 4) We have $\mathbb{X}_1 \cdot \mathbb{X}_2 = (1,2,-3)\cdot(1,1,1) = 1 + 2 - 3 = 0$.

To find a vector orthogonal to $\mathbb{X}_1$ we put:

$\mathbb{X}_1 \cdot \mathbb{X}_3 = (1,2,-3)\cdot(x,y,z) = x + 2y - 3z = 0 \Rightarrow x = 3z - 2y$.

To get the vector $\mathbb{X}_3$ to be orthogonal to $\mathbb{X}_2$ we put:

$\mathbb{X}_2 \cdot \mathbb{X}_3 = (1,1,1)\cdot(x,y,z) = (1,1,1)\cdot(3z-2y,y,z) = 3z - 2y + y + z = 0 \Rightarrow$

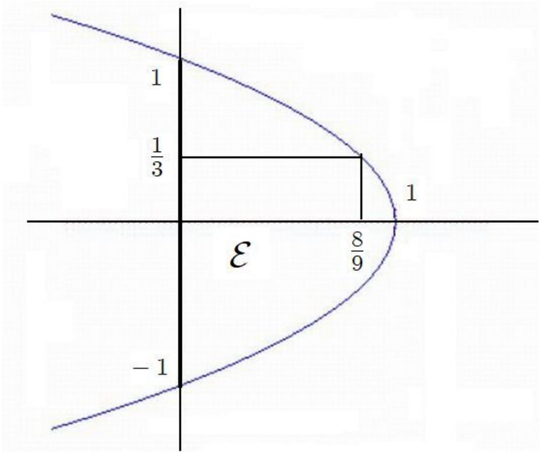
$\Rightarrow 4z - y = 0 \Rightarrow y = 4z$ and so $x = 3z - 2y = 3z - 8z = -5z$.

So $\mathbb{X}_3 = (-5z, 4z, z)$. If we choose $z = -1$ we get: $\mathbb{X}_3 = (5, -4, -1)$.

And so we get the basis $\mathbb{X}=\{\mathbb{X}_1=(1,2,-3);\mathbb{X}_2=(1,1,1);\mathbb{X}_3=(5,-4,-1)\}$.
To find the coordinates of the vector $\mathbb{Y}=(1,0,1)$ in the basis $\mathbb{X}$ we must solve the system:
$$\left\|\begin{matrix}1 & 1 & 5\\ 2 & 1 & -4\\ -3 & 1 & -1\end{matrix}\right\|\cdot\left\|\begin{matrix}x\\ y\\ z\end{matrix}\right\|=\left\|\begin{matrix}1\\ 0\\ 1\end{matrix}\right\|.$$
By elementary operation on the rows of the augmented matrix we get:
$$\left\|\begin{matrix}1 & 1 & 5 & | & 1\\ 2 & 1 & -4 & | & 0\\ -3 & 1 & -1 & | & 1\end{matrix}\right\|\Rightarrow\left\|\begin{matrix}1 & 1 & 5 & | & 1\\ 0 & -1 & -14 & | & -2\\ 0 & 4 & 14 & | & 4\end{matrix}\right\|\Rightarrow\left\|\begin{matrix}1 & 1 & 5 & | & 1\\ 0 & -1 & -14 & | & -2\\ 0 & 3 & 0 & | & 2\end{matrix}\right\|.$$
So we solve the system:
$$\begin{cases}x+y+5z=1\\ 0x-y-14z=-2\\ 0x+3y+0z=2\end{cases}\Rightarrow\begin{cases}x=1-y-5z=-\frac{1}{7}\\ z=\frac{1}{14}(2-y)=\frac{2}{21}\\ y=\frac{2}{3}\end{cases}.$$
The coordinates of the vector $\mathbb{Y}=(1,0,1)$ in the basis $\mathbb{X}$ are $\left(-\dfrac{1}{7},\dfrac{2}{3},\dfrac{2}{21}\right)$.

II M 1) For the system $\begin{cases}f(x,y,z)=x^2+y^2+z^3-3xy=0\\ g(x,y,z)=x^3+y^3-3z^2+xyz=0\end{cases}$ we get:
$\begin{cases}f(1,1,1)=1+1+1-3=0\\ g(1,1,1)=1+1-3+1=0\end{cases}$, so the system is satisfied at point $(1,1,1)$.
We construct the Jacobian matrix $\dfrac{\partial(f,g)}{\partial(x,y,z)}=\left\|\begin{matrix}2x-3y & 2y-3x & 3z^2\\ 3x^2+yz & 3y^2+xz & xy-6z\end{matrix}\right\|$ from which we get: $\dfrac{\partial(f,g)}{\partial(x,y,z)}(1,1,1)=\left\|\begin{matrix}-1 & -1 & 3\\ 4 & 4 & -5\end{matrix}\right\|$.
As $\left|\begin{matrix}-1 & 3\\ 4 & -5\end{matrix}\right|=5-12=-7\neq 0$ it is possible to define an implicit function like $x\to(y(x),z(x))$ and also an implicit function like $y\to(x(y),z(y))$.
As $\left|\begin{matrix}-1 & -1\\ 4 & 4\end{matrix}\right|=0$ it is not possible to define an implicit function like $z\to(x(z),y(z))$.
We choose $x\to(y(x),z(x))$ and we get:
$$\frac{dy}{dx}=-\frac{\left|\begin{matrix}-1 & 3\\ 4 & -5\end{matrix}\right|}{\left|\begin{matrix}-1 & 3\\ 4 & -5\end{matrix}\right|}=-\frac{-7}{-7}=-1;\ \frac{dz}{dx}=-\frac{\left|\begin{matrix}-1 & -1\\ 4 & 4\end{matrix}\right|}{\left|\begin{matrix}-1 & 3\\ 4 & -5\end{matrix}\right|}=-\frac{0}{-7}=0.$$

II M 2) Firstly we write the problem $\begin{cases}\text{Max/min } f(x,y)=x(y+1)\\ \text{u.c.: }\begin{cases}x+y^2\leq 1\\ 0\leq x\end{cases}\end{cases}$ in the form:
$$\begin{cases}\text{Max/min } f(x,y)=x(y+1)\\ \text{u.c.: }\begin{cases}x+y^2-1\leq 0\\ -x\leq 0\end{cases}\end{cases}.$$
The objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a compact set, and so maximum and minimum values surely exist. The constraints are qualified. Since $f(x,y)=x(y+1)\geq 0,\ \forall(x,y)\in\mathcal{E}$ and for all the points $(0,y)$ it is $f(0,y)=0$, all the points $\{(x,y): x=0,\ -1\leq y\leq 1\}$ are minimum points for the problem.



The Lagrangian function of the problem is:
$\Lambda(x, y, \lambda_1, \lambda_2) = x\,(y+1) - \lambda_1\big(x + y^2 - 1\big) - \lambda_2(-x)$.

1) case $\lambda_1 = 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = y + 1 = 0 \\ \Lambda'_y = x = 0 \\ x + y^2 \le 1 \\ 0 \le x \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \\ 0 \le 1 \\ 0 \le 0 \end{cases}$$: the point has already been studied.

2) case $\lambda_1 \ne 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = y + 1 - \lambda_1 = 0 \\ \Lambda'_y = x - 2\lambda_1 y = 0 \\ x + y^2 = 1 \\ 0 \le x \end{cases} \Rightarrow \begin{cases} y = \lambda_1 - 1 \\ x = 2\lambda_1(\lambda_1 - 1) = 2\lambda_1^2 - 2\lambda_1 \\ 2\lambda_1^2 - 2\lambda_1 + \lambda_1^2 + 1 - 2\lambda_1 = 1 \\ 0 \le x \end{cases} \Rightarrow \begin{cases} y = \lambda_1 - 1 \\ x = 2\lambda_1^2 - 2\lambda_1 \\ 3\lambda_1^2 - 4\lambda_1 = 0 \\ 0 \le x \end{cases}$$

$$\Rightarrow \begin{cases} x = 2\lambda_1^2 - 2\lambda_1 \\ y = \lambda_1 - 1 \\ \lambda_1(3\lambda_1 - 4) = 0 \\ 0 \le x \end{cases} \Rightarrow \begin{cases} x = 2\lambda_1^2 - 2\lambda_1 = 0 \\ y = \lambda_1 - 1 = -1 \\ \lambda_1 = 0 \\ 0 \le 0 \end{cases}$$ the point has already been studied

or $$\Rightarrow \begin{cases} x = 2\,\frac{16}{3} - 2\,\frac{4}{3} = \frac{8}{9} \\ y = \frac{1}{3} \\ \lambda_1 = \frac{4}{3} \\ 0 \le \frac{8}{9} \end{cases}$$; since $\lambda_1 = \dfrac{4}{3} > 0$ the point may be a maximum point.

3) case $\lambda_1 = 0, \lambda_2 \ne 0$:

$$\begin{cases} \Lambda'_x = y + 1 + \lambda_2 = 0 \\ \Lambda'_y = x = 0 \\ x = 0 \\ x + y^2 \le 1 \end{cases}$$: points $(0, y)$ have already been studied.

4) case $\lambda_1 \ne 0, \lambda_2 \ne 0$:

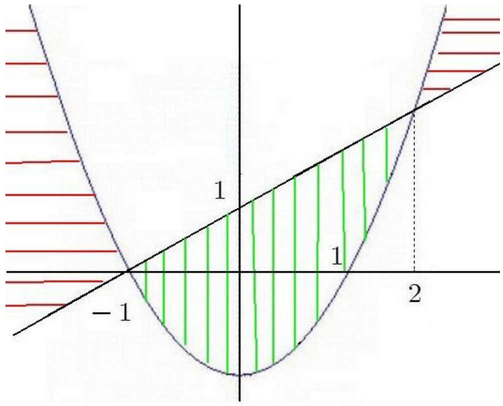
$$\begin{cases} \Lambda'_x = y + 1 - \lambda_1 + \lambda_2 = 0 \\ \Lambda'_y = x - 2\lambda_1 y = 0 \\ x + y^2 = 1 \\ x = 0 \end{cases}$$. Since the points solutions of $\begin{cases} x + y^2 = 1 \\ 0 = x \end{cases}$ are $(0, 1)$ and $(0, -1)$, such points have already been studied.

So the point $\left(\dfrac{8}{9}, \dfrac{1}{3}\right)$ is the maximum point, with $f\left(\dfrac{8}{9}, \dfrac{1}{3}\right) = \dfrac{32}{27}$ while all the points $\{(x, y) : x = 0, \ -1 \le y \le 1\}$ with $f(0, y) = 0$ are minimum points for the problem.

II M 3) For the existence field of the function $f(x,y)=\log\left(\dfrac{x-y+1}{y-x^2+1}\right)$ we must satisfy the condition: $\dfrac{x-y+1}{y-x^2+1}>0$ whose solution is given by the union of the solutions of the two inequality systems:

$$\begin{cases} x-y+1>0 \\ y-x^2+1>0 \end{cases} \Rightarrow \begin{cases} y<x+1 \\ y>x^2-1 \end{cases} \text{ or } \begin{cases} x-y+1<0 \\ y-x^2+1<0 \end{cases} \Rightarrow \begin{cases} y>x+1 \\ y<x^2-1 \end{cases}.$$

We graphically represent the situation as follows:



The vertical green lines indicate the solution of the first system, the horizontal red lines the solution of the second system.

For the gradient of the function we get:

$$\frac{\partial f}{\partial x}=\frac{y-x^2+1}{x-y+1}\cdot\frac{1(y-x^2+1)-(-2x)(x-y+1)}{(y-x^2+1)^2}=\frac{x^2-2xy+2x+y+1}{(x-y+1)(y-x^2+1)}$$

$$\frac{\partial f}{\partial y}=\frac{y-x^2+1}{x-y+1}\cdot\frac{-1(y-x^2+1)-1(x-y+1)}{(y-x^2+1)^2}=\frac{x^2-x-2}{(x-y+1)(y-x^2+1)}.$$

So $\nabla f(0,0)=(1,-2)$.

II M 4) The function $f(x,y)=k\left(x^2-y^2\right)+x-y$ is a polynomial, so it is twice differentiable. So, to calculate the first order directional derivatives we get:

$D_{e_1}f(1,0)=\nabla f(1,0)\cdot(1,0)=f'_x(1,0)$ and $D_{e_2}f(0,1)=\nabla f(0,1)\cdot(0,1)=f'_y(0,1)$.

Then from $D_{e_1}f(1,0)=D_{e_2}f(0,1)$ we get $f'_x(1,0)=f'_y(0,1)$.

Since: $f'_x(x,y)=2kx+1\Rightarrow f'_x(1,0)=2k+1$ and

$f'_y(x,y)=-2ky-1\Rightarrow f'_y(0,1)=-2k-1$ from $f'_x(1,0)=f'_y(0,1)$ we get:

$2k+1=-2k-1\Rightarrow 4k=-2\Rightarrow k=-\dfrac{1}{2}$.

The second order directional derivative it is obtained as:

$D^2_{e_1,e_2}f(1,1)=e_1\cdot\mathbb{H}(1,1)\cdot e_2^T$. Since $\mathbb{H}(x,y)=\begin{Vmatrix}2k & 0\\ 0 & -2k\end{Vmatrix}=\begin{Vmatrix}-1 & 0\\ 0 & 1\end{Vmatrix}$ we get:

$$D^2_{e_1,e_2}f(1,1)=\|1\quad 0\|\cdot\mathbb{H}(1,1)\cdot\begin{Vmatrix}0\\1\end{Vmatrix}=\|1\quad 0\|\cdot\begin{Vmatrix}-1 & 0\\ 0 & 1\end{Vmatrix}\cdot\begin{Vmatrix}0\\1\end{Vmatrix}=0.$$