TASK MATHEMATICS for ECONOMIC APPLICATIONS 24/3/2018

I M 1) $z = e^{1+4\pi i} \cdot e^{-1+3\pi i} = e^{1+(-1)+4\pi i+7\pi i} = e^{7\pi i} = (\cos 7\pi + i \sin 7\pi) =$
$= (\cos \pi + i \sin \pi) = -1$. And so:

$$\sqrt[3]{-1} = 1\left(\cos\left(\frac{\pi}{3} + k\,\frac{2\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + k\,\frac{2\pi}{3}\right)\right); 0 \le k \le 2 .$$

For $k = 0$: $\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}$;

For $k = 1$: $\cos \pi + i \sin \pi = -1$;

For $k = 2$: $\cos\left(\frac{5\pi}{3}\right) + i \sin\left(\frac{5\pi}{3}\right) = \frac{1}{2} - i\,\frac{\sqrt{3}}{2}$.

I M 2) $\begin{Vmatrix} x+1 & x & x \\ y & y+1 & y \\ -y & -y & 1-y \end{Vmatrix} \Rightarrow |\mathbb{A} - \lambda \mathbb{I}| = \begin{vmatrix} x+1-\lambda & x & x \\ y & y+1-\lambda & y \\ -y & -y & 1-y-\lambda \end{vmatrix} =$

$(C_1 \leftarrow C_1 - C_2)$ and $(C_3 \leftarrow C_3 - C_2)$

$$= \begin{vmatrix} 1-\lambda & x & 0 \\ \lambda-1 & y+1-\lambda & \lambda-1 \\ 0 & -y & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} y+1-\lambda & \lambda-1 \\ -y & 1-\lambda \end{vmatrix} - (\lambda-1)\begin{vmatrix} x & 0 \\ -y & 1-\lambda \end{vmatrix} =$$

$= (1-\lambda)[(y+1-\lambda)(1-\lambda) - (\lambda-1)(-y) + x(1-\lambda)] =$
$= (1-\lambda)(1-\lambda)[(y+1-\lambda) - y + x] = (1-\lambda)(1-\lambda)(1-\lambda+x)$. So the matrix admits the eigenvalue $\lambda = 1$ with algebraic multiplicity equal to 3 if $x = 0, \forall y$.

I M 3) From $f(x_1, x_2, x_3) = (ax_1, x_1 + bx_2, x_1 + x_2 + cx_3)$ we get, in matrix form:

$$\mathbb{A}_{3,3} \cdot \mathbb{X}_{3,1} = \mathbb{Y}_{3,1} \Rightarrow \begin{Vmatrix} a & 0 & 0 \\ 1 & b & 0 \\ 1 & 1 & c \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} y_1 \\ y_2 \\ y_3 \end{Vmatrix} .$$

From $f(1,1,1) = (0,1,2)$ we get:

$$\begin{Vmatrix} a & 0 & 0 \\ 1 & b & 0 \\ 1 & 1 & c \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 1 \\ 2 \end{Vmatrix} \Rightarrow \begin{cases} a = 0 \\ 1 + b = 1 \\ 1 + 1 + c = 2 \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = 0 \\ c = 0 \end{cases} . \text{ So } \mathbb{A} = \begin{Vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{Vmatrix} .$$

Since $\text{Rank}(\mathbb{A}) = 2$, we get $\text{Dim}(\text{Imm}(\mathbb{A})) = 2$ and $\text{Dim}(\text{Ker}(\mathbb{A})) = 3 - 2 = 1$.
To find a basis for the Image we apply the Rouchè-Capelli Theorem to the augmented matrix $\begin{Vmatrix} 0 & 0 & 0 & | & y_1 \\ 1 & 0 & 0 & | & y_2 \\ 1 & 1 & 0 & | & y_3 \end{Vmatrix}$. We get $\text{Rank}(\mathbb{A}) = \text{Rank}(\mathbb{A}|\mathbb{Y})$ if $y_1 = 0$. So the vectors belonging to the Image are the vectors of the form $(0, y_2, y_3)$. A basis for the Image may be:
$\mathbb{W} = \{(0,1,0); (0,0,1)\}$. To find a basis for the Kernel we solve the homogeneous system:

$$\begin{Vmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} 0 = 0 \\ x_1 = 0 \\ x_1 + x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ \forall x_3 \end{cases}$$

. So the vectors belonging to the Kernel are the vectors of the form $(0,0,x)$. A basis for the Kernel is $\mathbb{W} = \{(0,0,1)\}$.

I M 4) From $\mathbb{B} = \begin{Vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 2 & 1 \end{Vmatrix}$ we get $\mathbb{B}^T = \begin{Vmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{Vmatrix}$ and so:

$$\mathbb{A}+\mathbb{B}^T=\begin{Vmatrix}1&1&1\\0&1&1\\0&0&1\end{Vmatrix}+\begin{Vmatrix}1&2&4\\0&1&2\\0&0&1\end{Vmatrix}=\begin{Vmatrix}2&3&5\\0&2&3\\0&0&2\end{Vmatrix}.$$

For the determinant we get $\left|\mathbb{A}+\mathbb{B}^T\right|=\begin{vmatrix}2&3&5\\0&2&3\\0&0&2\end{vmatrix}=2\cdot 2\cdot 2=8\,.$

Now we calculate the adjoint matrix and we get: $\operatorname{Adj}\left(\mathbb{A}+\mathbb{B}^T\right)=\begin{Vmatrix}4&0&0\\-6&4&0\\-1&-6&4\end{Vmatrix}.$

Now we calculate the transpose of the adjoint matrix and we get: $\begin{Vmatrix}4&-6&-1\\0&4&-6\\0&0&4\end{Vmatrix}.$

Finally, dividing by the determinant we get the inverse matrix:

$$\left(\mathbb{A}+\mathbb{B}^T\right)^{-1}=\begin{Vmatrix}\frac{1}{2}&-\frac{3}{4}&-\frac{1}{8}\\0&\frac{1}{2}&-\frac{3}{4}\\0&0&\frac{1}{2}\end{Vmatrix}.$$

II M 1) From the equation $f(x,y,z)=xe^{y-z}-ye^{z-x}+ze^{x-y}=0$ we get: $f(0,0,0)=0-0+0=0$ and so the point $(0,0,0)$ satisfies the equation. Then $\nabla f(x,y,z)=(e^{y-z}+ye^{z-x}+ze^{x-y},xe^{y-z}-e^{z-x}-ze^{x-y},-xe^{y-z}-ye^{z-x}+e^{x-y})$ and so we get $\nabla f(0,0,0)=(1,-1,1)$. Since $f'_z=1\neq 0$ there exists an implicit function $(x,y)\to z(x,y)$ with partial derivatives :

$\frac{\partial z}{\partial x}(0,0)=-\frac{1}{1}=-1$ and $\frac{\partial z}{\partial y}(0,0)=-\frac{-1}{1}=1\,.$

For the equation of the tangent plan at $(0,0)$ we get:

$z-0=-1\,(x-0)+1\,(y-0)\Rightarrow z=-x+y\Rightarrow x-y+z=0\,.$

II M 2) To solve the problem $\begin{cases}\text{Max/min } f(x,y)=x^2-y^2\\ \text{u.c.: } x^2+4y^2\le 4\end{cases}$ we observe that objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is an ellipse, a compact set, and so maximum and minimum values surely exist. The constraints are qualified. The Lagrangian function of the problem is: $\Lambda(x,y,\lambda)=x^2-y^2-\lambda\left(x^2+4y^2-4\right)$.

1) case $\lambda=0$:

$\begin{cases}\Lambda'_x=2x=0\\ \Lambda'_y=-2y=0\\ x^2+4y^2\le 4\end{cases}\Rightarrow\begin{cases}x=0\\ y=0\\ 0\le 4\end{cases}$. Since $\mathbb{H}(x,y)=\mathbb{H}(0,0)=\begin{Vmatrix}2&0\\0&-2\end{Vmatrix}$ the point $(0,0)$ is a saddle point.

2) case $\lambda\neq 0$:

$\begin{cases}\Lambda'_x=2x-2\lambda x=2x(1-\lambda)=0\\ \Lambda'_y=-2y-8\lambda y=-2y(1+4\lambda)=0\\ x^2+4y^2=4\end{cases}\Rightarrow\begin{cases}x=0\\ y=0\\ 0\neq 4\end{cases}$ unacceptable solution;

$\cup\begin{cases}x=0\\ \lambda=-\frac{1}{4}\\ 4y^2=4\end{cases}\Rightarrow\begin{cases}x=0\\ \lambda=-\frac{1}{4}\\ y=\pm 1\end{cases}$; since $\lambda<0$ these points may be minimum points;

$\cup \begin{cases} \lambda = 1 \\ y = 0 \\ x^2 = 4 \end{cases} \Rightarrow \begin{cases} \lambda = 1 \\ y = 0 \\ x = \pm 2 \end{cases}$; since $\lambda > 0$ these points may be maximum points;

$\cup \begin{cases} \lambda = 1 \\ \lambda = -\frac{1}{4} \\ x^2 + 4y^2 = 4 \end{cases}$ there is no solution.

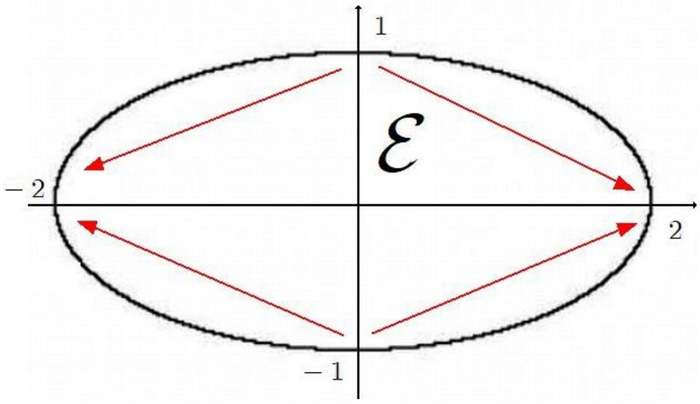
Since $f(0,1) = f(0,-1) = -1$ while $f(2,0) = f(-2,0) = 4$ we see that $(0,1)$ and $(0,-1)$ are minimum points while $(2,0)$ and $(-2,0)$ are maximum points.

If we want to study our problem in the boundary points, only for exercise, since we have just completely solved the problem, we can use the parametric form for the ellipse:

$\begin{cases} x = 2\cos t \\ y = \sin t \end{cases} \Rightarrow f(x.y) \to f(t) = 4\cos^2 t - \sin^2 t = 5\cos^2 t - 1$. By deriving we get:

$f'(t) = 10\cos t \cdot (-\sin t) = -10\cos t \sin t = -2\sin 2t$.

Hence $f'(t) \geq 0 \Rightarrow \sin 2t \leq 0 \Rightarrow \left\{\frac{\pi}{2} \leq t \leq \pi\right\} \cup \left\{\frac{3\pi}{2} \leq t \leq 2\pi\right\}$ and so we get the results that we have already found.



II M 3) Since $f(x,y) = x\,e^y + y\,e^x$ is a twice differentiable function, we get:

$\mathcal{D}_v f(0,0) = \nabla f(0,0) \cdot v$ and $\mathcal{D}^2_{v,v} f(0,0) = v \cdot \mathbb{H}(0,0) \cdot v^T$.

$\nabla f(x,y) = (e^y + y\,e^x, x\,e^y + e^x) \Rightarrow \nabla f(0,0) = (1,1)$.

$\mathcal{D}_v f(0,0) = \nabla f(0,0) \cdot v = (1,1) \cdot (\cos\alpha, \sin\alpha) = \cos\alpha + \sin\alpha = 0$ from which we get : $\cos\alpha = -\sin\alpha \Rightarrow \alpha = \frac{3}{4}\pi$ and $\alpha = \frac{7}{4}\pi$.

Since $\mathbb{H}(x,y) = \begin{Vmatrix} y\,e^x & e^y + e^x \\ e^y + e^x & x\,e^y \end{Vmatrix} \Rightarrow \mathbb{H}(0,0) = \begin{Vmatrix} 0 & 2 \\ 2 & 0 \end{Vmatrix}$ we get:

$\mathcal{D}^2_{v,v} f(0,0) = \|\cos\alpha \;\; \sin\alpha\| \cdot \begin{Vmatrix} 0 & 2 \\ 2 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} \cos\alpha \\ \sin\alpha \end{Vmatrix} = \|\cos\alpha \;\; \sin\alpha\| \cdot \begin{Vmatrix} 2\sin\alpha \\ 2\cos\alpha \end{Vmatrix} =$

$= 2\sin\alpha\cos\alpha + 2\sin\alpha\cos\alpha = 2\sin 2\alpha$.

II M 4) From $f : \mathbb{R} \to \mathbb{R}^3, t \to \left(\sin(t^2), e^{2t}, \log(1+t)\right)$ we get:

$\mathbb{X}(t) = \left(\sin(t^2), e^{2t}, \log(1+t)\right)$ from which $\mathbb{X}(0) = (0,1,0)$. Then:

$\mathbb{X}'(t) = \left(2t\cos(t^2), 2\,e^{2t}, \frac{1}{1+t}\right)$ from which $\mathbb{X}'(0) = (0,2,1)$.

The equation of the tangent line to this curve at the point $t = 0$ is:

$t \to \mathbb{X}(0) + t \cdot \mathbb{X}'(0) = (0,1,0) + t \cdot (0,2,1) \Rightarrow t \to (0, 1+2t, t)$.