TASK MATHEMATICS for ECONOMIC APPLICATIONS 11/6/2018

I M 1) $z = \frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} = \frac{1-1+2i-(1-1-2i)}{1-i^2} =$
$= \frac{4i}{2} = 2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$
So $\sqrt{z} = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + k\pi\right) + i\sin\left(\frac{\pi}{4} + k\pi\right)\right),\ 0 \le k \le 1\,.$
For $k = 0\,:\ \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right) = 1 + i\,;$
For $k = 1\,:\ \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right) = -1 - i\,.$

I M 2) From $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{Vmatrix}$ we get :

$$|\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & -1 \\ 0 & 0 & -1 & -1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ \lambda & 1-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & \lambda & -1-\lambda \end{vmatrix} =$$

$(C_1 \leftarrow C_1 - C_2$ and $C_3 \leftarrow C_3 - C_4)$ and then $(R_1 \leftarrow R_1 + R_2$ and $R_4 \leftarrow R_4 + R_3)$

$$= \begin{vmatrix} 0 & 2-\lambda & 0 & 0 \\ \lambda & 1-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & 0 & -2-\lambda \end{vmatrix} = (\lambda - 2)\begin{vmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & 0 & -2-\lambda \end{vmatrix} =$$

$= (\lambda - 2)\,\lambda\,(-\lambda)(-2-\lambda) = 0$ for $\lambda_1 = \lambda_2 = 0;\ \lambda_3 = 2;\ \lambda_4 = -2\,.$
To find the eigenspace corresponding to the eigenvalue $\lambda = 0$ we solve the system:

$$\|\mathbb{A} - 0\cdot\mathbb{I}\| \cdot \mathbb{X} = \mathbb{A}\cdot\mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \\ w \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}.$$

Since $\text{Rank}(\mathbb{A} - 0\cdot\mathbb{I}) = \text{Rank}(\mathbb{A}) = 2$ the dimension of the eigenspace corresponding to the eigenvalue $\lambda = 0$ is $m_0^g = 4 - \text{Rank}(\mathbb{A} - 0\cdot\mathbb{I}) = 4 - \text{Rank}(\mathbb{A}) = 4 - 2 = 2\,.$ The system becomes $\begin{cases} x + y = 0 \\ -z - w = 0 \end{cases} \Rightarrow \begin{cases} y = -x \\ w = -z \end{cases}$ and the eigenvectors corresponding to the eigenvalue $\lambda = 0$ are $\mathbb{V} = (x, -x, z, -z),\ x, z \in \mathbb{R}\,.$ A basis for the eigenspace may be: $\mathbb{V}_0 = \{(1, -1, 0, 0); (0, 0, 1, -1)\}\,.$
To find the eigenspace corresponding to the eigenvalue $\lambda = 2$ we solve the system:

$$\|\mathbb{A} - 2\cdot\mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & -1 & -3 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \\ w \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix}.$$

Since $\lambda = 2$ is a simple eigenvalue, we get $m_2^g = m_2^a = 1$ and the system becomes:

$\begin{cases} x-y=0 \\ 3z+w=0 \\ z+3w=0 \end{cases} \Rightarrow \begin{cases} y=x \\ z=0 \\ w=0 \end{cases}$. The eigenvectors corresponding to the eigenvalue $\lambda=2$ are $\mathbb{V}=(x,x,0,0),\ x\in\mathbb{R}$. A basis for the eigenspace may be the vector: $\mathbb{V}_2=\{(1,1,0,0)\}$.
To find the eigenspace corresponding to the eigenvalue $\lambda=-2$ we solve the system:

$$\|\mathbb{A}+2\cdot\mathbb{I}\|\cdot\mathbb{X}=\mathbb{O}\Rightarrow\left\|\begin{matrix}3&1&0&0\\1&3&0&0\\0&0&1&-1\\0&0&-1&1\end{matrix}\right\|\cdot\left\|\begin{matrix}x\\y\\z\\w\end{matrix}\right\|=\left\|\begin{matrix}0\\0\\0\\0\end{matrix}\right\|.$$

Since $\lambda=-2$ is a simple eigenvalue, we get $m^g_{-2}=m^a_{-2}=1$ and the system becomes:
$\begin{cases} 3x+y=0 \\ x+3y=0 \\ z-w=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ z=w \end{cases}$. The eigenvectors corresponding to the eigenvalue $\lambda=-2$ are $\mathbb{V}=(0,0,z,z),\ z\in\mathbb{R}$. A basis for the eigenspace may be the vector: $\mathbb{V}_{-2}=\{(0,0,1,1)\}$.

I M 3) Since the dimension of the Kernel of the map is equal to 1, we get that $\text{Rank}(\mathbb{A})=2$, and so the determinant of the matrix must be equal to zero: $|\mathbb{A}|=0$:

$$|\mathbb{A}|=\begin{vmatrix}1&0&1\\0&k&-1\\1&-1&2\end{vmatrix}=\begin{vmatrix}1&0&1\\0&k&-1\\0&-1&1\end{vmatrix}=1\cdot(k-1)=0\Rightarrow k=1.$$ So:

$$\mathbb{A}=\left\|\begin{matrix}1&0&1\\0&1&-1\\1&-1&2\end{matrix}\right\|\Rightarrow|\mathbb{A}-\lambda\mathbb{I}|=\begin{vmatrix}1-\lambda&0&1\\0&1-\lambda&-1\\1&-1&2-\lambda\end{vmatrix}=$$

$$=(1-\lambda)[(1-\lambda)(2-\lambda)-1]+1\,(\lambda-1)=(1-\lambda)\big(\lambda^2-3\lambda+2-1-1\big)=$$

$=(1-\lambda)\big(\lambda^2-3\lambda\big)=\lambda(1-\lambda)(\lambda-3)=0$. So the matrix has three simple eigenvalues and it is a diagonalizable one.

I M 4) To solve the problem we can apply Rouchè-Capelli Theorem. So we study the augmented matrix: $(\mathbb{A}|\mathbb{Y})=\left\|\begin{matrix}1&-1&1&-1&|&-1\\1&0&1&1&|&1\\0&-1&m&-2&|&k\end{matrix}\right\|$. By elementary operations on the rows ($R_2\leftarrow R_2-R_1$ and then $R_3\leftarrow R_3+R_2$) we get:

$$(\mathbb{A}|\mathbb{Y})\Rightarrow\left\|\begin{matrix}1&-1&1&-1&|&-1\\0&1&0&2&|&2\\0&-1&m&-2&|&k\end{matrix}\right\|\Rightarrow\left\|\begin{matrix}1&-1&1&-1&|&-1\\0&1&0&2&|&2\\0&0&m&0&|&k+2\end{matrix}\right\|.$$ So:

For $m\neq 0,\ \forall k$ we get: $\text{Rank}(\mathbb{A})=\text{Rank}(\mathbb{A}|\mathbb{Y})=3$: we have ∞^1 solutions;
For $m=0$ and $k=-2$ we get: $\text{Rank}(\mathbb{A})=\text{Rank}(\mathbb{A}|\mathbb{Y})=2$: we have ∞^2 solutions;
For $m=0$ and $k\neq-2$ we get: $\text{Rank}(\mathbb{A})=2<\text{Rank}(\mathbb{A}|\mathbb{Y})=3$: we have no solutions; it is impossible to express the vector $\mathbb{Y}$ with a linear combination of the vectors $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$ and $\mathbb{X}_4$.

II M 1) From the equation $f(x,y,z)=x^3+y^3+z^3+3xy+3yz=1$ we get:
$f(1,1,-1)=1+1-1+3-3=1$ and so the point $(1,1,-1)$ satisfies the equation.
Then $\nabla f(x,y,z)=\big(3x^2+3y,3y^2+3x+3z,3z^2+3y\big)$ and $\nabla f(1,1,-1)=(6,3,6)$.
Since $f'_z=6\neq0$ there exists an implicit function $(x,y)\to z(x,y)$ with partial derivatives : $\dfrac{\partial z}{\partial x}(1,1)=-\dfrac{6}{6}=-1$ and $\dfrac{\partial z}{\partial y}(1,1)=-\dfrac{3}{6}=-\dfrac{1}{2}$.

II M 2) To solve the problem $\begin{cases} \text{Max/min } f(x,y) = x^4 + y^4 \\ \text{u.c.: } x^2 + y^2 \le 1 \end{cases}$ we observe that objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a circle, so a compact set, and so maximum and minimum values surely exist. The constraints are qualified.
The Lagrangian function of the problem is: $\Lambda(x,y,\lambda) = x^4 + y^4 - \lambda\big(x^2 + y^2 - 1\big)$.
1) case $\lambda = 0$:

$$\begin{cases} \Lambda'_x = 4x^3 = 0 \\ \Lambda'_y = 4y^3 = 0 \\ x^2 + y^2 \le 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \le 1 \end{cases}. \ \mathbb{H}(x,y) = \begin{Vmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{Vmatrix} \text{ and } \mathbb{H}(0,0) = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix},$$

By the Hessian $\mathbb{H}(0,0)$ we haven't any information about the point $(0,0)$, but trivialy we can see that $(0,0)$ is a minimum point since $f(0,0) = 0$ and $f(x,y) \ge 0, \forall\,(x,y)$.

2) case $\lambda \ne 0$:

$$\begin{cases} \Lambda'_x = 4x^3 - 2\lambda x = 2x(2x^2 - \lambda) = 0 \\ \Lambda'_y = 4y^3 - 2\lambda y = 2y(2y^2 - \lambda) = 0 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \ne 1 \end{cases} \text{ unacceptable solution;}$$

$$\cup \begin{cases} x = 0 \\ \lambda = 2 \\ y^2 = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ \lambda = 2 \\ y = \pm 1 \end{cases}; \text{ since } \lambda > 0 \text{ these points may be maximum points;}$$

$$\cup \begin{cases} \lambda = 2 \\ y = 0 \\ x^2 = 1 \end{cases} \Rightarrow \begin{cases} \lambda = 2 \\ y = 0 \\ x = \pm 1 \end{cases}; \text{ since } \lambda > 0 \text{ these points may be maximum points;}$$

$$\cup \begin{cases} x^2 = \dfrac{\lambda}{2} \\ y^2 = \dfrac{\lambda}{2} \\ \dfrac{\lambda}{2} + \dfrac{\lambda}{2} = 1 \end{cases} \Rightarrow \begin{cases} x = \pm\dfrac{\sqrt{2}}{2} \\ y = \pm\dfrac{\sqrt{2}}{2} \\ \lambda = 1 \end{cases}; \text{ since } \lambda > 0 \text{ these points may be maximum points.}$$

If we want to complete the study of our problem in the boundary points, we can use the parametric form for the circle:

$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \Rightarrow f(x.y) \to f(t) = \cos^4 t + \sin^4 t$. By deriving we get:

$f'(t) = 4\cos^3 t \cdot (-\sin t) + 4\sin^3 t \cdot \cos t = -4\sin t \cdot \cos t \cdot \big(\cos^2 t - \sin^2 t\big)$.

And so $f'(t) = -2\sin 2t \cdot \cos 2t = -\sin 4t$.

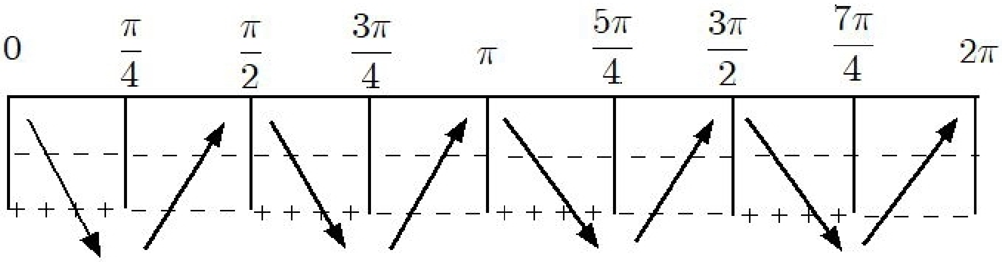
Since $\sin t \ge 0$ for $0 \le t \le \pi$ we get: $\sin 4t \ge 0$ for:

$$\left(0 \le t \le \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2} \le t \le \frac{3\pi}{4}\right) \cup \left(\pi \le t \le \frac{5\pi}{4}\right) \cup \left(\frac{3\pi}{2} \le t \le \frac{7\pi}{4}\right).$$

So $f'(t) \ge 0$ for:

$$\left(\frac{\pi}{4} \le t \le \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{4} \le t \le \pi\right) \cup \left(\frac{5\pi}{4} \le t \le \frac{3\pi}{2}\right) \cup \left(\frac{7\pi}{4} \le t \le 2\pi\right).$$

So the four points $\left(\pm\dfrac{\sqrt{2}}{2}, \pm\dfrac{\sqrt{2}}{2}\right)$ are not maximum points.



II M 3) $f(x,y)=\log\left(e^{x^2+y^2}+e^{x^2-y^2}\right)$ is a differentiable function $\forall\,(x,y)\in\mathbb{R}^2$.

From $\nabla f(x,y)=\left(\dfrac{2xe^{x^2+y^2}+2xe^{x^2-y^2}}{e^{x^2+y^2}+e^{x^2-y^2}},\dfrac{2ye^{x^2+y^2}-2ye^{x^2-y^2}}{e^{x^2+y^2}+e^{x^2-y^2}}\right)$: $\nabla f(0,0)=(0,0)$

and $\nabla f(-1,1)=\left(\dfrac{-2e^2-2e^0}{e^2+e^0},\dfrac{2e^2-2e^0}{e^2+e^0}\right)=\left(-2,2\,\dfrac{e^2-1}{e^2+1}\right)$. And so:

$$D_v f(0,0)=\nabla f(0,0)\cdot v=(0,0)\cdot\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)=0\,;$$

$$\mathcal{D}_w f(-1,1)=\nabla f(-1,1)\cdot w=\left(-2,2\,\frac{e^2-1}{e^2+1}\right)\cdot\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)=\frac{2\sqrt{2}}{e^2+1}\,.$$

II M 4) $f(x,y)=(x-y)^2+(y-3x)^2$ is a differentiable function $\forall\,(x,y)\in\mathbb{R}^2$.

$f(x,y)=(x-y)^2+(y-3x)^2=10\,x^2+2y^2-8xy\Rightarrow\nabla f(x,y)=(20x-8y,4y-8x)$.

$I\,OC$: $\begin{cases}20x-8y=0\\-8x+4y=0\end{cases}\Rightarrow\begin{cases}x=0\\y=0\end{cases}$. We have only one stationary point: $(0,0)$.

$\mathbb{H}f(x,y)=\mathbb{H}f(0,0)=\begin{Vmatrix}20 & -8\\-8 & 4\end{Vmatrix}$.

$II\,OC$: $\begin{cases}|\mathbb{H}_1|=20>0 \text{ or } |\mathbb{H}_1|=4>0\\|\mathbb{H}_2|=16>0\end{cases}\Rightarrow(0,0)$ is a minimum point.