Prova Intermedia di Analisi Matematica del 13/11/2018 $\boxed{AM1}$

IM1) $z = \dfrac{e^{\log\sqrt{2} + i\frac{\pi}{4}}}{1-i} = \dfrac{e^{\log\sqrt{2}}\cdot(\cos\frac{\pi}{4} + i\,\mathrm{sen}\frac{\pi}{4})}{\sqrt{2}\cdot(\cos\frac{7}{4}\pi + i\,\mathrm{sen}\frac{7}{4}\pi)} = \dfrac{\sqrt{2}}{\sqrt{2}}\cdot\left(\cos\left(\frac{\pi}{4}-\frac{7}{4}\pi\right) + i\,\mathrm{sen}\left(\frac{\pi}{4}-\frac{7}{4}\pi\right)\right) =$

$= \cos\left(-\frac{3}{2}\pi\right) + i\,\mathrm{sen}\left(-\frac{3}{2}\pi\right) = \cos\frac{\pi}{2} + i\,\mathrm{sen}\frac{\pi}{2} = i.$

$\sqrt[3]{z} = \sqrt[3]{i} = \cos\left(\frac{\pi}{6} + k\cdot\frac{2\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{\pi}{6} + k\frac{2\pi}{3}\right);\ 0 \le k \le 2.$

Per $k=0$: $\cos\frac{\pi}{6} + i\,\mathrm{sen}\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}$; per $k=1$: $\cos\frac{5}{6}\pi + i\,\mathrm{sen}\frac{5}{6}\pi = -\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}$;

per $k=2$: $\cos\frac{3}{2}\pi + i\,\mathrm{sen}\frac{3}{2}\pi = -i$.

IM2) $f(x;y) = \begin{cases} \dfrac{x^2|y| + y^2|x|}{x^2+y^2} & : (x;y) \ne (0;0) \\ 0 & : (x;y) = (0;0) \end{cases}$. Per la continuità:

$\lim_{(x;y)\to(0;0)} \dfrac{x^2|y|+y^2|x|}{x^2+y^2} \Rightarrow \lim_{\rho\to 0} \dfrac{\rho^3\cdot(\cos^2\vartheta|\mathrm{sen}\vartheta| + \mathrm{sen}^2\vartheta|\cos\vartheta|)}{\rho^2} = 0$ in modo uniforme

in quanto $\left|\rho\cdot(\cos^2\vartheta|\mathrm{sen}\vartheta| + \mathrm{sen}^2\vartheta|\cos\vartheta|) - 0\right| \le \rho\cdot 2 < \varepsilon$ per $\rho < \frac{\varepsilon}{2} \Rightarrow f(x;y) \in \mathcal{C}(0;0)$.

$\dfrac{\partial f}{\partial x}(0;0) = \lim_{h\to 0}\left(\dfrac{h^2\cdot|0| + 0\cdot|h|}{h^2+0}\right)\cdot\dfrac{1}{h} = 0$; $\dfrac{\partial f}{\partial y}(0;0) = \lim_{h\to 0}\left(\dfrac{0\cdot|h| + h^2\cdot|0|}{0+h^2}\right)\cdot\dfrac{1}{h} = 0$.

Per la differenziabilità: $\lim_{(x;y)\to(0;0)}\left(\dfrac{x^2|y|+y^2|x|}{x^2+y^2} - 0 - (0;0)(x;y)\right)\cdot\dfrac{1}{\sqrt{x^2+y^2}} \Rightarrow$

$\Rightarrow \lim_{\rho\to 0}\dfrac{\rho^3(|\mathrm{sen}\vartheta| + |\cos\vartheta|)}{\rho^2}\cdot\dfrac{1}{\rho} = |\mathrm{sen}\vartheta| + |\cos\vartheta| \ne 0$: f non è differenziabile.

IM3) $f(x;y) = x e^{y-x} + y e^{x-y}$: funzione differenziabile 2 volte.

$\nabla f(x;y) = \left((1-x)e^{y-x} + y e^{x-y};\ x e^{y-x} + (1-y)e^{x-y}\right)$; $\nabla f(0;0) = (1;1)$.

$\mathcal{D}_v f(0;0) = \nabla f(0;0)\cdot(\cos\alpha;\mathrm{sen}\alpha) = (1;1)(\cos\alpha;\mathrm{sen}\alpha) = \cos\alpha + \mathrm{sen}\alpha = 0 \Rightarrow$

$\Rightarrow \mathrm{sen}\alpha = -\cos\alpha \Rightarrow \alpha = \frac{3}{4}\pi$.

$H(x;y) = \begin{Vmatrix} (x-2)e^{y-x} + y e^{x-y} & (1-x)e^{y-x} + (1-y)e^{x-y} \\ (1-x)e^{y-x} + (1-y)e^{x-y} & x e^{y-x} + (y-2)e^{x-y} \end{Vmatrix}$; $H(0;0) = \begin{Vmatrix} -2 & 2 \\ 2 & -2 \end{Vmatrix}$

$\mathcal{D}^2_{v,-v} f(0;0) = \begin{Vmatrix}\cos\alpha & -\cos\alpha\end{Vmatrix}\begin{Vmatrix} -2 & 2 \\ 2 & -2\end{Vmatrix}\cdot\begin{Vmatrix}-\cos\alpha \\ \cos\alpha\end{Vmatrix} = \begin{Vmatrix}\cos\alpha & -\cos\alpha\end{Vmatrix}\begin{Vmatrix} 4\cos\alpha \\ -4\cos\alpha\end{Vmatrix} = +4\cos^2\alpha + 4\cos^2\alpha =$

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$= +8\cos^2\alpha$ e se $\alpha = \frac{3}{4}\pi \Rightarrow \mathcal{D}^2_{v_1 - v} f(0;0) = +8 \cdot \frac{1}{2} = +4.$

IM4) $f(x;y) = y e^{x^2+x} + x e^{y-y^2} = 0$; $f(-1;1) = 1 \cdot e^0 + (-1) e^0 = 0.$

$\nabla f(x;y) = \left(y(2x+1) e^{x^2+x} + e^{y-y^2} ; \; e^{x^2+x} + x(1-2y) e^{y-y^2} \right)$; $\nabla f(-1;1) = (0;2)$

Si può definire $x \to y(x)$ con $y'(-1) = -\frac{0}{2} = 0.$

$H(x;y) = \begin{Vmatrix} y[2+(2x+1)^2] \cdot e^{x^2+x} & (2x+1) e^{x^2+x} + (1-2y) e^{y-y^2} \\ (2x+1) e^{x^2+x} + (1-2y) e^{y-y^2} & x[-2+(1-2y)^2] e^{y-y^2} \end{Vmatrix}$; $H(-1;1) = \begin{Vmatrix} 3 & -2 \\ -2 & 1 \end{Vmatrix}.$

$$y''(-1) = -\frac{f''_{xx} + 2 f''_{xy} \cdot y'(-1) + f''_{yy} \cdot [y'(-1)]^2}{f'_y} = -\frac{3 + 2 \cdot (-2) \cdot 0 + 1 \cdot 0^2}{2} = -\frac{3}{2}.$$

Dato che $y''(-1) = -\frac{3}{2} < 0 \Rightarrow -1$ è un punto di massimo.

IM5) $\begin{cases} f(x;y;z) = x \log y - y \log z + xz = 0 \\ g(x;y;z) = x^3 y - y^3 z + x z^3 + 1 = 0 \end{cases} \Rightarrow \begin{cases} f(0;1;1) = 0 \\ g(0;1;1) = -1+1 = 0 \end{cases}$

$$\frac{\partial(f;g)}{\partial(x;y;z)} = \begin{Vmatrix} \log y + z & x \cdot \frac{1}{y} - \log z & -y \cdot \frac{1}{z} + x \\ 3x^2 y + z^3 & x^3 - 3y^2 z & -y^3 + 3xz^2 \end{Vmatrix}$$

$$\frac{\partial(f;g)}{\partial(x;y;z)}(0;1;1) = \begin{Vmatrix} 0+1 & 0-0 & -1+0 \\ 0+1 & 0-3 & -1+0 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & -1 \\ 1 & -3 & -1 \end{Vmatrix}.$$

Dato che $\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0$ non si può definire $y \to (x(y); z(y))$.

Essendo $\begin{vmatrix} 0 & -1 \\ -3 & -1 \end{vmatrix} = -3 \neq 0$ si può definire $x \to (y(x); z(x))$ con

$$\frac{dy}{dx} = -\frac{\begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} 0 & -1 \\ -3 & -1 \end{vmatrix}} = -\frac{0}{-3} = 0; \quad \frac{dz}{dx} = -\frac{\begin{vmatrix} 0 & 1 \\ -3 & 1 \end{vmatrix}}{\begin{vmatrix} 0 & -1 \\ -3 & -1 \end{vmatrix}} = -\frac{3}{-3} = 1.$$

Equazione retta tangente: $x \to (y_0; z_0) + x \cdot X'(x_0) \Rightarrow$

$x \to (1;1) + x(0;1) = (1; 1+x).$