TASK MATHEMATICS for ECONOMIC APPLICATIONS
15/01/2019

I M 1) If ρ and ρ' are respectively the modulus of the complex number z and the modulus of one of its cubic roots, and if α and α' are their arguments, it is $\rho' = \sqrt[3]{\rho}$ and $\alpha' = \dfrac{\alpha}{3}$.

Thus $\rho = (\rho')^3 = 3^3 = 27\,,\ \alpha = 3\,\alpha' = \dfrac{3\,\pi}{12} = \dfrac{\pi}{4}$ and so:

$z = 27\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right) = \dfrac{27}{\sqrt{2}}\,(1+i)$. For the cubic roots of z we have:

$$\sqrt[3]{z} = \sqrt[3]{27\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)} = 3\left(\cos\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{12} + \frac{2k\pi}{3}\right)\right)$$

with $k = 0, 1, 2$. The three roots are:

$z_1 = 3\left(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right) = \dfrac{3}{2}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right)$,

$z_2 = 3\left(\cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4}\right) = -\dfrac{3}{2}\sqrt{2}(1-i)$ and

$z_3 = 3\left(\cos\dfrac{17\pi}{12} + i\sin\dfrac{17\pi}{12}\right) = -\dfrac{3}{2}\left(\sqrt{2-\sqrt{3}} + i\sqrt{2+\sqrt{3}}\right)$.

I M 2) The characteristic polynomial of $\mathbb{A}$ is $p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 1-\lambda & -1 \\ 1 & k-\lambda \end{vmatrix} =$

$= (1-\lambda)(k-\lambda) + 1 = \lambda^2 - (k+1)\lambda + k + 1$.

The discriminant is $\Delta = (k+1)^2 - 4(k+1) = (k+1)(k-3)$.

When $\Delta > 0 \Rightarrow k < -1$ or $k > 3$ the matrix $\mathbb{A}$ has two real and distinct eigenvalues:

$\lambda_{1,2} = \dfrac{(k+1) \pm \sqrt{(k+1)(k-3)}}{2}$;

when $\Delta = 0 \Rightarrow k = -1$ or $k = 3$ the matrix $\mathbb{A}$ has two real and equal eigenvalues:

$\lambda_{1,2} = 0$ for $k = -1$ and $\lambda_{1,2} = 2$ for $k = 3$;

when $\Delta < 0 \Rightarrow -1 < k < 3$ the matrix $\mathbb{A}$ has two complex conjugated eigenvalues:

$\lambda_{1,2} = \dfrac{k+1 \pm i\sqrt{-(k+1)(k-3)}}{2}$.

I M 3) The three vectors $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ belongin to $\mathbb{R}^3$ form a basis if and only if the determinant of the matrix $\|\,\mathbb{X}_1 \quad \mathbb{X}_2 \quad \mathbb{X}_3\,\| = \left\|\begin{matrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{matrix}\right\|$ is different from zero.

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = 1\cdot\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + (-1)\cdot\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = 2 + 2 = 4 \neq 0.$$

So $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ form a basis for $\mathbb{R}^3$. If α, β and γ are the coordinates of the vector $\mathbb{Y}$ in the base $\{\mathbb{X}_1, \mathbb{X}_2, \mathbb{X}_3\}$ it is $\mathbb{Y} = \alpha\mathbb{X}_1 + \beta\mathbb{X}_2 + \gamma\mathbb{X}_3$, which leads to the system:

$$\begin{cases} \alpha + \beta + 2\gamma = 1 \\ \alpha - \beta = 3 \\ \alpha - \gamma = 2 \end{cases} \Rightarrow \begin{cases} \alpha + \alpha - 3 + 2\alpha - 4 = 1 \\ \beta = \alpha - 3 \\ \gamma = \alpha - 2 \end{cases} \Rightarrow \begin{cases} \alpha = 2 \\ \beta = -1 \\ \gamma = 0 \end{cases}.$$

Since the third coordinate is $\gamma = 0$, this implies that the vector $\mathbb{Y}$ belongs to the plane spanned by the vectors $\mathbb{X}_1$ and $\mathbb{X}_2$.

I M 4) For a linear map $f(\mathbb{X}) = \mathbb{A} \cdot \mathbb{X}$ the dimension of the Image is equal to the rank of the matrix $\mathbb{A}$ while the dimension of the Kernel is equal to the difference between the dimension of the domain and the dimension of the Image.
To calculate the rank of $\mathbb{A}$ we reduce the matrix by elementary operations on its lines:
By $(R_3 \leftarrow R_3 - R_1 - R_2)$ and then $(R_2 \leftarrow R_2 - R_1)$ we get:

$$\left\|\begin{matrix} 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & m & 1 & 1 & k \end{matrix}\right\| \to \left\|\begin{matrix} 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & m & 0 & 0 & k \end{matrix}\right\| \to \left\|\begin{matrix} 1 & -1 & 0 & 1 & -1 \\ 0 & 2 & 1 & -1 & 2 \\ 0 & m & 0 & 0 & k \end{matrix}\right\|.$$

From the last matrix we can conclude that Rank$(\mathbb{A})$ is equal to 3 if $m \neq 0$ or $k \neq 0$, if $m = 0$ and $k = 0$ Rank$(\mathbb{A})$ is equal to 2.

So $\text{Dim(Imm)} = \begin{cases} 2 & \text{if } m = k = 0 \\ 3 & \text{otherwise} \end{cases} \Rightarrow \text{Dim(Ker)} = \begin{cases} 3 & \text{if } m = k = 0 \\ 2 & \text{otherwise} \end{cases}.$

The dimension of the Image is minimal when $m = 0$ and $k = 0$.
The vector $\mathbb{Y} = (2, -1, 2)$, by Rouchè-Capelli Theorem, belongs to Imm(f) if and only if Rank$(\mathbb{A})$ = Rank$(\mathbb{A}|\mathbb{Y})$.
By elementary operations on $(\mathbb{A}|\mathbb{Y})$, $(R_3 \leftarrow R_3 - R_1 - R_2)$ and $(R_2 \leftarrow R_2 - R_1)$ we get:

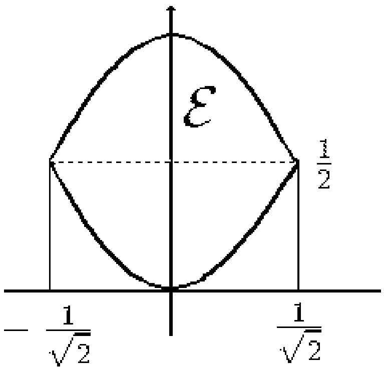
$$\left\|\begin{matrix} 1 & -1 & 0 & 1 & -1 & 2 \\ 1 & 1 & 1 & 0 & 1 & -1 \\ 2 & 0 & 1 & 1 & 0 & 2 \end{matrix}\right\| \to \left\|\begin{matrix} 1 & -1 & 0 & 1 & -1 & 2 \\ 0 & 2 & 1 & -1 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{matrix}\right\|.$$

So we see that Rank$(\mathbb{A}) = 2$ while Rank$(\mathbb{A}|\mathbb{Y}) = 3$ so $\mathbb{Y}$ doesn't belong to Imm(f).

II M 1) From the equation $f(x, y, z) = x \sin y - y \cos z + x^2 z = 0$ we get:
$f(1, 0, 0) = 0 - 0 + 0 = 0$ and so the point $(1, 0, 0)$ satisfies the equation. Then :
$\nabla f(x, y, z) = \left(\sin y + 2xz, x \cos y - \cos z, y \sin z + x^2\right)$ and $\nabla f(1, 0, 0) = (0, -1, 0)$.
Since only $f'_y(1, 0, 0) \neq 0$ it is possible to define only an implicit function $(x, z) \to y$ whose derivatives are: $\dfrac{\partial y}{\partial x}(1, 0) = -\dfrac{0}{-1} = 0$ and $\dfrac{\partial y}{\partial z}(1, 0) = -\dfrac{0}{-1} = 0$.

II M 2) To solve the problem $\begin{cases} \text{Max/min } f(x, y) = x + y \\ \text{u.c. } x^2 \leq y \leq 1 - x^2 \end{cases}$ firstly we write the problem in the form: $\begin{cases} \text{Max/min } f(x, y) = x + y \\ \text{u.c.} \begin{cases} x^2 - y \leq 0 \\ x^2 + y - 1 \leq 0 \end{cases} \end{cases}$; then we observe that the objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a compact set, and so maximum and minimum values surely exist. The constraints are qualified.

$$\begin{cases} y = x^2 \\ y = 1 - x^2 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{2}} \\ y = \frac{1}{2} \end{cases} \cup \begin{cases} x = -\frac{1}{\sqrt{2}} \\ y = \frac{1}{2} \end{cases}.$$



The Lagrangian function is: $\Lambda(x,y,\lambda)=x+y-\lambda_1\left(x^2-y\right)-\lambda_2\left(x^2+y-1\right)$.

1) case $\lambda_1=0,\lambda_2=0$:

$\begin{cases}\Lambda'_x=1\neq 0\\ \Lambda'_y=1\neq 0\\ x^2\leq y\\ y\leq 1-x^2\end{cases}$: no solutions.

2) case $\lambda_1\neq 0,\lambda_2=0$:

$\begin{cases}\Lambda'_x=1-2\lambda_1 x=0\\ \Lambda'_y=1+\lambda_1=0\\ y=x^2\\ y\leq 1-x^2\end{cases}\Rightarrow\begin{cases}x=-\frac{1}{2}\\ \lambda_1=-1\\ y=\frac{1}{4}\\ \frac{1}{4}\leq\frac{3}{4}\text{ : true}\end{cases}$. Since $\lambda_1<0$ the point $\left(-\dfrac{1}{2},\dfrac{1}{4}\right)$ may be a minimum point.

3) case $\lambda_1=0,\lambda_2\neq 0$:

$\begin{cases}\Lambda'_x=1-2\lambda_2 x=0\\ \Lambda'_y=1-\lambda_2=0\\ y=1-x^2\\ x^2\leq y\end{cases}\Rightarrow\begin{cases}x=\frac{1}{2}\\ \lambda_2=1\\ y=\frac{3}{4}\\ \frac{1}{4}\leq\frac{3}{4}\text{ : true}\end{cases}$. Since $\lambda_2>0$ the point $\left(\dfrac{1}{2},\dfrac{3}{4}\right)$ may be a maximum point.

4) case $\lambda_1\neq 0,\lambda_2\neq 0$:

$\begin{cases}\Lambda'_x=1-2\lambda_1 x-2\lambda_2 x=0\\ \Lambda'_y=1+\lambda_1-\lambda_2=0\\ y=x^2\\ y=1-x^2\end{cases}$. The points solutions of $\begin{cases}y=x^2\\ y=1-x^2\end{cases}$ are $\left(\frac{1}{\sqrt{2}},\frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}},\frac{1}{2}\right)$ and so we get the systems:

$$\begin{cases}1-2\lambda_1\frac{1}{\sqrt{2}}-2\lambda_2\frac{1}{\sqrt{2}}=0\\ \Lambda'_y=1+\lambda_1-\lambda_2=0\\ x=\frac{1}{\sqrt{2}}\\ y=\frac{1}{2}\end{cases}\Rightarrow\begin{cases}1-\sqrt{2}-2\sqrt{2}\lambda_1=0\\ \lambda_2=1+\lambda_1\\ x=\frac{1}{\sqrt{2}}\\ y=\frac{1}{2}\end{cases}\Rightarrow\begin{cases}\lambda_1=\frac{1-\sqrt{2}}{2\sqrt{2}}\\ \lambda_2=\frac{\sqrt{2}+1}{2\sqrt{2}}\\ x=\frac{1}{\sqrt{2}}\\ y=\frac{1}{2}\end{cases}.$$

Since $\lambda_1<0$ and $\lambda_2>0$ the point $\left(\dfrac{1}{\sqrt{2}},\dfrac{1}{2}\right)$ is nor a maximum nor a minimum point.

$$\begin{cases} 1+2\lambda_1\frac{1}{\sqrt{2}}+2\lambda_2\frac{1}{\sqrt{2}}=0 \\ \Lambda'_y=1+\lambda_1-\lambda_2=0 \\ x=-\frac{1}{\sqrt{2}} \\ y=\frac{1}{2} \end{cases} \Rightarrow \begin{cases} 1+\sqrt{2}+2\sqrt{2}\lambda_1=0 \\ \lambda_2=1+\lambda_1 \\ x=\frac{1}{\sqrt{2}} \\ y=\frac{1}{2} \end{cases} \Rightarrow \begin{cases} \lambda_1=-\frac{1+\sqrt{2}}{2\sqrt{2}} \\ \lambda_2=\frac{\sqrt{2}-1}{2\sqrt{2}} \\ x=\frac{1}{\sqrt{2}} \\ y=\frac{1}{2} \end{cases}.$$

Since $\lambda_1 < 0$ and $\lambda_2 > 0$ the point $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ is nor a maximum nor a minimum point.

So, from Weierstrass Theorem, $\left(\frac{1}{2}, \frac{3}{4}\right)$ is the maximum point with $f\left(\frac{1}{2}, \frac{3}{4}\right) = \frac{5}{4}$ while $\left(-\frac{1}{2}, \frac{1}{4}\right)$ is the minimum point with $f\left(-\frac{1}{2}, \frac{3}{4}\right) = \frac{1}{4}$.

II M 3) Since the function $f(x,y) = x\,e^{y-x}$ is clearly a differentiable function $\forall\,(x,y) \in \mathbb{R}^2$, we simply calculate $D_v f(1,1) = \nabla f(1,1) \cdot v$.
Since $\nabla f(x,y) = (e^{y-x} - x\,e^{y-x}, x\,e^{y-x}) \Rightarrow \nabla f(1,1) = (0,1)$. And so:
$D_v f(1,1) = \nabla f(1,1) \cdot v = (0,1) \cdot (\cos\alpha, \sin\alpha) = \sin\alpha = 0$ for $\alpha = 0$ and $\alpha = \pi$.

II M 4) We study the leading principal minors of the matrix $\mathbb{H} = \begin{Vmatrix} k & 0 \\ 0 & k-1 \end{Vmatrix}$.
The leading principal minors of the first order are:
$|\mathbb{H}_1| = k \geq 0$ for $k \geq 0$; $|\mathbb{H}_1| = k - 1 \geq 0$ for $k \geq 1$.
The leading principal minor of the second order is:
$|\mathbb{H}_2| = k \cdot (k-1) \geq 0$ for $k \leq 0$ or $k \geq 1$.
And so:
For $k < 0$: $|\mathbb{H}_1| < 0$ and $|\mathbb{H}_2| > 0$: the point is a Maximum point;
For $k > 1$: $|\mathbb{H}_1| > 0$ and $|\mathbb{H}_2| > 0$: the point is a Minimum point;
For $0 < k < 1$: $|\mathbb{H}_2| < 0$: the point is a Saddle point;
For $k = 0$ and $k = 1$ we have a semidefinite quadratic form and so we cannot decide.