Compito di Analisi Matematica del 12/2/2019 $\boxed{\text{CAM1}}$

IM1) $x^6+x^2=x^2(x^4+1)=0 \Rightarrow x^2=0 \Rightarrow x_1=x_2=0;\ x^4=-1 \Rightarrow x=\sqrt[4]{-1}$.

$-1=\cos\pi+i\,\text{sen}\,\pi \Rightarrow \sqrt[4]{-1}=\cos\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)+i\,\text{sen}\left(\frac{\pi}{4}+k\cdot\frac{\pi}{2}\right);\ 0\le k\le 3.$

$\underline{k=0}$: $\cos\frac{\pi}{4}+i\,\text{sen}\frac{\pi}{4}=\frac{1}{\sqrt2}+i\frac{1}{\sqrt2}$; $\underline{k=1}$: $\cos\frac{3}{4}\pi+i\,\text{sen}\frac{3}{4}\pi=-\frac{1}{\sqrt2}+i\frac{1}{\sqrt2}$;

$\underline{k=2}$: $\cos\frac{5}{4}\pi+i\,\text{sen}\frac{5}{4}\pi=-\frac{1}{\sqrt2}-i\frac{1}{\sqrt2}$; $\underline{k=3}$: $\cos\frac{7}{4}\pi+i\,\text{sen}\frac{7}{4}\pi=\frac{1}{\sqrt2}-i\frac{1}{\sqrt2}$.

Somma radici: $0+0+\left(\frac{1}{\sqrt2}+i\frac{1}{\sqrt2}\right)+\left(-\frac{1}{\sqrt2}+i\frac{1}{\sqrt2}\right)+\left(-\frac{1}{\sqrt2}-i\frac{1}{\sqrt2}\right)+\left(\frac{1}{\sqrt2}-i\frac{1}{\sqrt2}\right)=0.$

IM2) Continuità: $\lim\limits_{(x;y)\to(0;0)}\frac{x^3-y^3}{x^2+y^2} \Rightarrow \lim\limits_{\rho\to0}\frac{\rho^{\cancel{3}}(\cos^3\vartheta-\text{sen}^3\vartheta)}{\cancel{\rho^2}}=0$ in modo uniforme in quanto $\rho\left|\cos^3\vartheta-\text{sen}^3\vartheta\right|\le 2\rho$ quindi $f(x;y)\in\mathcal{C}(0;0)$.

$\frac{\partial f}{\partial x}(0;0)=\lim\limits_{h\to0}\left(\frac{h^3-0}{h^2+0}-0\right)\cdot\frac{1}{h}=\lim\limits_{h\to0}\frac{h^3}{h^3}=1$;

$\frac{\partial f}{\partial y}(0;0)=\lim\limits_{h\to0}\left(\frac{0-h^3}{0+h^2}-0\right)\cdot\frac{1}{h}=\lim\limits_{h\to0}\left(-\frac{h^3}{h^3}\right)=-1.$

Differenziabilità: $\lim\limits_{(x;y)\to(0;0)}\left(\frac{x^3-y^3}{x^2+y^2}-0-(1;-1)\cdot(x-0;y-0)\right)\cdot\frac{1}{\sqrt{x^2+y^2}}=$

$=\lim\limits_{(x;y)\to(0;0)}\frac{x^3-y^3-(x-y)(x^2+y^2)}{(x^2+y^2)\sqrt{x^2+y^2}}=\lim\limits_{(x;y)\to(0;0)}\frac{\cancel{x^3}-\cancel{y^3}-\cancel{x^3}-xy^2+yx^2+\cancel{y^3}}{(x^2+y^2)\sqrt{x^2+y^2}}=$

$=\lim\limits_{(x;y)\to(0;0)}\frac{xy(x-y)}{(x^2+y^2)\sqrt{x^2+y^2}} \Rightarrow \lim\limits_{\rho\to0}\frac{\rho^3\cos\vartheta\,\text{sen}\,\vartheta(\cos\vartheta-\text{sen}\,\vartheta)}{\rho^2\cdot\rho}=\cos\vartheta\,\text{sen}\,\vartheta(\cos\vartheta-\text{sen}\,\vartheta).$

La funzione non è differenziabile in $(0;0)$.

IM3) $\begin{cases} f(x;y;z;w)=xe^y-ye^z-ze^w+we^x=0\\ g(x;y;z;w)=xyz-yzw-xzw+xyw=0\end{cases} \Rightarrow \begin{cases} f(1;1;1;1)=e-e-e+e=0\\ g(1;1;1;1)=1-1-1+1=0\end{cases}.$

$\frac{\partial(f;g)}{\partial(x;y;z;w)}=\left\|\begin{matrix} e^y+we^x & xe^y-e^z & -ye^z-e^w & -ze^w+e^x\\ yz-zw+yw & xz-zw+xw & xy-yw-xw & -yz-xz+xy\end{matrix}\right\|$

$\frac{\partial(f;g)}{\partial(x;y;z;w)}(1;1;1;1)=\left\|\begin{matrix} 2e & 0 & -2e & 0\\ 1 & 1 & -1 & -1\end{matrix}\right\|.$

$\boxed{CAM2}$

Dato che $\begin{vmatrix} -2e & 0 \\ -1 & -1 \end{vmatrix} = 2e \neq 0$ si può definire una funzione $(x;y) \to (z(x;y);w(x;y))$.

$$\frac{\partial z}{\partial x} = -\frac{\begin{vmatrix} 2e & 0 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} -2e & 0 \\ -1 & -1 \end{vmatrix}} = -\frac{-2e}{2e} = 1;\ \frac{\partial z}{\partial y} = -\frac{\begin{vmatrix} 0 & 0 \\ 1 & -1 \end{vmatrix}}{\begin{vmatrix} -2e & 0 \\ -1 & -1 \end{vmatrix}} = 0;\ \frac{\partial w}{\partial x} = -\frac{\begin{vmatrix} -2e & 2e \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} -2e & 0 \\ -1 & -1 \end{vmatrix}} = 0;\ \frac{\partial w}{\partial y} = -\frac{\begin{vmatrix} -2e & 0 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} -2e & 0 \\ -1 & -1 \end{vmatrix}} = -\frac{-2e}{2e} = 1.$$

I M4) $f(x;y) = xe^{y-x} - ye^{x-y}$; funzione differenziabile $\forall (x;y) \in \mathbb{R}^2$.

$$\nabla f(x;y) = \left(e^{y-x} - xe^{y-x} - ye^{x-y};\ xe^{y-x} - e^{x-y} + ye^{x-y}\right) = \left((1-x)e^{y-x} - ye^{x-y};\ xe^{y-x} + (y-1)e^{x-y}\right).$$

$\mathcal{D}_v f(0;0) = \nabla f(0;0) \cdot (\cos\alpha; \operatorname{sen}\alpha) = (1;-1)(\cos\alpha;\operatorname{sen}\alpha) = \cos\alpha - \operatorname{sen}\alpha$;

$\mathcal{D}_v f(1;1) = \nabla f(1;1) \cdot (\cos\alpha; \operatorname{sen}\alpha) = (-1;1)(\cos\alpha;\operatorname{sen}\alpha) = \operatorname{sen}\alpha - \cos\alpha$.

$\mathcal{D}_v f(0;0) = \mathcal{D}_v f(1;1) \Rightarrow \cos\alpha - \operatorname{sen}\alpha = \operatorname{sen}\alpha - \cos\alpha \Rightarrow \cos\alpha = \operatorname{sen}\alpha \Rightarrow \alpha = \frac{\pi}{4};\ \alpha = \frac{5}{4}\pi$.

II M1) $\begin{cases} \text{Max/min } f(x;y) = xy^2 - x^2 - y^2 \\ \text{s.v. } x^2+y^2 \leq 1 \end{cases}$ Funzione continua e differenziabile; $\mathcal{E}$ limitato e chiuso, vincolo qualificato.

$\Lambda(x;y;\lambda) = xy^2 - x^2 - y^2 - \lambda(x^2+y^2-1)$.

Caso $\lambda = 0$ $\begin{cases} \Lambda'_x = y^2 - 2x = 0 \\ \Lambda'_y = 2xy - 2y = 2y(x-1) = 0 \\ x^2+y^2 \leq 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0+0 \leq 1 \end{cases} \cup \begin{cases} x = 1 \\ y^2 = 2 \\ 1+2 \leq 1 \end{cases}$ $(1;\sqrt{2})$ e $(1;-\sqrt{2}) \notin \mathcal{E}$.

vera falsa

$H(x;y) = \begin{Vmatrix} -2 & 2y \\ 2y & 2x-2 \end{Vmatrix} \Rightarrow H(0;0) = \begin{Vmatrix} -2 & 0 \\ 0 & -2 \end{Vmatrix} \Rightarrow \begin{cases} -2 < 0 \\ 4 > 0 \end{cases} \Rightarrow (0;0)$ è punto di Massimo.

Caso $\lambda \neq 0$ $\begin{cases} \Lambda'_x = y^2 - 2x - 2\lambda x = 0 \\ \Lambda'_y = 2xy - 2y - 2\lambda y = 2y(x-1-\lambda) = 0 \\ x^2+y^2 = 1 \end{cases} \Rightarrow \begin{cases} 2x(1+\lambda) = 0 \\ y = 0 \\ x^2 = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda = -1 \end{cases} \cup \begin{cases} x = -1 \\ y = 0 \\ \lambda = -1 \end{cases}$

Min? Min?

$\cup \begin{cases} x = 1+\lambda \\ y^2 = 2x(1+\lambda) = 2(1+\lambda)^2 \\ (1+\lambda)^2 + 2(1+\lambda)^2 = 1 \end{cases} \Rightarrow \begin{cases} x = 1+\lambda \\ y^2 = 2(1+\lambda)^2 \\ (1+\lambda)^2 = \frac{1}{3} \end{cases} \Rightarrow \begin{cases} x = 1+\lambda \\ y^2 = 2(1+\lambda)^2 \\ 1+\lambda = \pm\frac{1}{\sqrt{3}} \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{3}} \\ y^2 = \frac{2}{3} \\ \lambda = -1+\frac{1}{\sqrt{3}} < 0 \end{cases} \cup \begin{cases} x = -\frac{1}{\sqrt{3}} \\ y^2 = \frac{2}{3} \\ \lambda = -1-\frac{1}{\sqrt{3}} < 0. \end{cases}$

Abbiamo quattro soluzioni: $\begin{cases} x = \frac{1}{\sqrt{3}} \\ y = \frac{\sqrt{2}}{\sqrt{3}} \\ \lambda = -1+\frac{1}{\sqrt{3}} < 0 \end{cases}$; $\begin{cases} x = \frac{1}{\sqrt{3}} \\ y = -\frac{\sqrt{2}}{\sqrt{3}} \\ \lambda = -1+\frac{1}{\sqrt{3}} < 0 \end{cases}$; $\begin{cases} x = -\frac{1}{\sqrt{3}} \\ y = \frac{\sqrt{2}}{\sqrt{3}} \\ \lambda = -1-\frac{1}{\sqrt{3}} < 0 \end{cases}$; $\begin{cases} x = -\frac{1}{\sqrt{3}} \\ y = -\frac{\sqrt{2}}{\sqrt{3}} \\ \lambda = -1-\frac{1}{\sqrt{3}} < 0 \end{cases}$.

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CAM 3

Analisi sulla frontiera di $\mathcal{E}$ ponendo $(x;y) = (\cos t; \operatorname{sen} t)$.

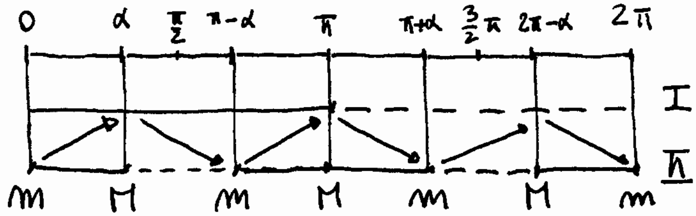
$f(t) = \cos t \cdot \operatorname{sen}^2 t - \cos^2 t - \operatorname{sen}^2 t = \cos t \cdot \operatorname{sen}^2 t - 1 \Rightarrow f'(t) = -\operatorname{sen}^3 t + 2 \operatorname{sen} t \cos^2 t \geq 0$

$\operatorname{sen} t(2\cos^2 t - \operatorname{sen}^2 t) = \operatorname{sen} t(3\cos^2 t - 1) \geq 0$.

I) $\operatorname{sen} t \geq 0$ per $0 \leq t \leq \pi$

II) $\cos^2 t \geq \frac{1}{3}$ per $\cos t \leq -\frac{1}{\sqrt{3}} \cup \cos t \geq \frac{1}{\sqrt{3}}$

Sia α : $\cos\alpha = \frac{1}{\sqrt{3}}$.



In $(1;0)$ abbiamo un punto di Minimo; Nulla in $(-1;0)$ per indicazione contraria alla precedente.

In $\left(\frac{1}{\sqrt{3}}; \frac{\sqrt{2}}{\sqrt{3}}\right)$ e $\left(\frac{1}{\sqrt{3}}; -\frac{\sqrt{2}}{\sqrt{3}}\right)$ abbiamo una indicazione contraria alla precedente;

In $\left(-\frac{1}{\sqrt{3}}; \frac{\sqrt{2}}{\sqrt{3}}\right)$ e $\left(-\frac{1}{\sqrt{3}}; -\frac{\sqrt{2}}{\sqrt{3}}\right)$ abbiamo una indicazione concorde con la precedente.

Quindi $(0;0)$ è il punto di Massimo $(f(0;0) = 0)$ mentre $\left(-\frac{1}{\sqrt{3}}; \frac{\sqrt{2}}{\sqrt{3}}\right)$ e $\left(-\frac{1}{\sqrt{3}}; -\frac{\sqrt{2}}{\sqrt{3}}\right)$ sono punti di minimo con $f\left(-\frac{1}{\sqrt{3}}; \frac{\sqrt{2}}{\sqrt{3}}\right) = f\left(-\frac{1}{\sqrt{3}}; -\frac{\sqrt{2}}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}} - 1$.

II M2) $\begin{cases} x' = y + t^2 \\ y' = t - x \end{cases} \Rightarrow \begin{cases} x' - y = t^2 \\ x + y' = t \end{cases} \Rightarrow \begin{Vmatrix} D & -1 \\ 1 & D \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \end{Vmatrix} = \begin{Vmatrix} t^2 \\ t \end{Vmatrix}$.

$\begin{vmatrix} D & -1 \\ 1 & D \end{vmatrix} (x) = \begin{vmatrix} t^2 & -1 \\ t & D \end{vmatrix} \Rightarrow (D^2+1)(x) = 2t + t = 3t \Rightarrow x'' + x = 3t$.

$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$.

Soluzione omogenea: $x(t) = c_1 \operatorname{sen} t + c_2 \cos t$.

Per la soluzione della non omogenea pongo $x_0(t) = at + b \Rightarrow x_0'(t) = a$; $x_0''(t) = 0$.

Quindi $0 + at + b = 3t \Rightarrow a = 3$ e $b = 0$. Quindi $x(t) = c_1 \operatorname{sen} t + c_2 \cos t + 3t$.

$y(t) = x' - t^2 = c_1 \cos t - c_2 \operatorname{sen} t + 3 - t^2$. Quindi: $\begin{cases} x(t) = c_1 \operatorname{sen} t + c_2 \cos t + 3t \\ y(t) = c_1 \cos t - c_2 \operatorname{sen} t + 3 - t^2. \end{cases}$

$\boxed{\text{CAM4}}$

II M3) $y''' - y = 0 \Rightarrow y''' - y = 0 \Rightarrow \lambda^3 - 1 = 0 \Rightarrow \lambda = \sqrt[3]{1}$.

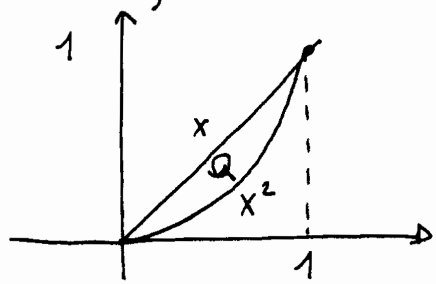
$1 = \cos 0 + i\,\text{sen}\,0 \Rightarrow \sqrt[3]{1} = \left(\cos\left(k\cdot\frac{2\pi}{3}\right) + i\,\text{sen}\left(k\cdot\frac{2\pi}{3}\right)\right)$; $0 \leq k \leq 2$.

per $k=0$: $\cos 0 + i\,\text{sen}\,0 = 1$; per $k=1$: $\cos\frac{2}{3}\pi + i\,\text{sen}\frac{2}{3}\pi = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$;

per $k=2$: $\cos\frac{4}{3}\pi + i\,\text{sen}\frac{4}{3}\pi = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. $\lambda_1 = 1$; $\lambda_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$; $\lambda_3 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

Soluzione generale omogenea: $y(x) = c_1 e^{x} + c_2 e^{-\frac{1}{2}x}\cdot\text{sen}\frac{\sqrt{3}}{2}x + c_3 e^{-\frac{1}{2}x}\cdot\cos\frac{\sqrt{3}}{2}x$.

II M4) $\iint_Q x\cdot\frac{\text{sen}\,y}{y}\,dx\,dy$; $Q = \left\{(x;y) \in \mathbb{R}^2 : 0 \leq x \leq 1 ; x^2 \leq y \leq x\right\}$.



La funzione presenta in $(0;0)$ una discontinuità che può essere eliminata. Dato che $\lim_{y\to 0}\frac{\text{sen}\,y}{y} = 1$, se poniamo $f(0;0) = 0\cdot 1 = 0$ la funzione diviene continua. Come si vede facilmente, se leggiamo la regione Q come normale rispetto all'asse x non si riesce a calcolare la primitiva rispetto a y. Quindi calcoliamo l'integrale come:

$$\int_0^1\left(\int_y^{\sqrt{y}}\frac{\text{sen}\,y}{y}\cdot x\,dx\right)dy = \int_0^1\frac{\text{sen}\,y}{y}\cdot\left(\frac{x^2}{2}\right|_y^{\sqrt{y}}dy = \int_0^1\frac{\text{sen}\,y}{y}\cdot\left(\frac{y}{2} - \frac{y^2}{2}\right)dy =$$

$$= \frac{1}{2}\int_0^1 \text{sen}\,y - y\cdot\text{sen}\,y\,dy = \frac{1}{2}\left[\left(-\cos y\right|_0^1 - \left(\left(-y\cos y\right|_0^1 - \int_0^1 -\cos y\,dy\right)\right] =$$

$$= \frac{1}{2}\left[\left(-\cos 1 + 1\right) - \left(-\cos 1 + 0\right) + \left(-\text{sen}\,y\right|_0^1\right] = \frac{1}{2}\left(1 - \text{sen}\,1\right).$$