TASK MATHEMATICS for ECONOMIC APPLICATIONS 12/02/2019

I M 1) Since $x^6+x^2=x^2\left(x^4+1\right)=0$ we get immediatly two roots: $x_1=x_2=0$.
The remaining four roots are obtained solving $x^4+1=0 \Rightarrow x=\sqrt[4]{-1}$.
From $-1=\cos\pi+i\sin\pi$ we get:
$\sqrt[4]{-1}=\cos\left(\frac{\pi}{4}+k\,\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{4}+k\,\frac{\pi}{2}\right),\ 0\le k\le 3$.
For $k=0$: $x_3=\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}+i\,\frac{1}{\sqrt{2}}$,
For $k=1$: $x_4=\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}+i\,\frac{1}{\sqrt{2}}$,
For $k=2$: $x_5=\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}=-\frac{1}{\sqrt{2}}-i\,\frac{1}{\sqrt{2}}$,
For $k=3$: $x_6=\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}=\frac{1}{\sqrt{2}}-i\,\frac{1}{\sqrt{2}}$.
So $\sum_{i=1}^{6}x_i=0$.

I M 2) The characteristic polynomial of $\mathbb{A}$ is :
$$p_{\mathbb{A}}(\lambda)=|\mathbb{A}-\lambda\mathbb{I}|=\begin{vmatrix}1-\lambda & 4 & 3\\ 2 & 3-\lambda & 1\\ 0 & 0 & k-\lambda\end{vmatrix}=(k-\lambda)[(1-\lambda)(3-\lambda)-8]=$$
$=(k-\lambda)\left(\lambda^2-4\lambda-5\right)=0$. From $\lambda^2-4\lambda-5=0$ we get:
$\lambda_{1,2}=2\pm\sqrt{4+5}=2\pm 3\Rightarrow\lambda_1=-1,\lambda_2=5$.
So we have multiple eigenvalues for $k=-1$ and for $k=5$.
To find m^g_{-1} we find the rank of the matrix $\|\mathbb{A}-(-1)\,\mathbb{I}\|=\begin{Vmatrix}2 & 4 & 3\\ 2 & 4 & 1\\ 0 & 0 & 0\end{Vmatrix}$;
since $\text{Rank}\left(\|\mathbb{A}+\mathbb{I}\|\right)=2$ we get $m^g_{-1}=3-2=1\le 2=m^a_{-1}$ and the matrix is not diagonalizable.
To find m^g_5 we find the rank of the matrix $\|\mathbb{A}-5\,\mathbb{I}\|=\begin{Vmatrix}-4 & 4 & 3\\ 2 & -2 & 1\\ 0 & 0 & 0\end{Vmatrix}$;
since $\text{Rank}\left(\|\mathbb{A}-5\,\mathbb{I}\|\right)=2$ we get $m^g_5=3-2=1\le 2=m^a_5$ and the matrix is not diagonalizable.

I M 3) From Rouchè-Capelli Theorem, if $\text{Rank}\,(\mathbb{A})=\text{Rank}\,(\mathbb{A}|\mathbb{Y})=k$ the system has ∞^{n-k} solutions, where n is the number of the variables; in our problem $n=4$ and so, to get ∞^2 solutions, we need $k=2$. We study the Rank of the augmented matrix:
$\|\mathbb{A}|\mathbb{Y}\|=\begin{Vmatrix}1 & -1 & -1 & 1 & | & 1\\ 2 & 1 & 2 & 1 & | & -1\\ 3 & 0 & 1 & h & | & m\end{Vmatrix}$. By elementary operations on the rows:
$(R_3\leftarrow R_3-R_1-R_2)$ we get: $\begin{Vmatrix}1 & -1 & -1 & 1 & | & 1\\ 2 & 1 & 2 & 1 & | & -1\\ 0 & 0 & 0 & h-2 & | & m\end{Vmatrix}$.
So $\text{Rank}\,(\mathbb{A})=\text{Rank}\,(\mathbb{A}|\mathbb{Y})=2$ if and only if $h=2$ and $m=0$.

So we have to solve the system $\begin{cases} x_1 - x_2 - x_3 + x_4 = 1 \\ 2x_1 + x_2 + 2x_3 + x_4 = -1 \\ 3x_1 + x_3 + 2x_4 = 0 \end{cases}$ whose augmented matrix is:

$$\|\mathbb{A}|\mathbb{Y}\| = \left\| \begin{matrix} 1 & -1 & -1 & 1 & | & 1 \\ 2 & 1 & 2 & 1 & | & -1 \\ 3 & 0 & 1 & 2 & | & 0 \end{matrix} \right\|.$$

By elementary operations: $(R_2 \leftarrow R_2 - 2R_1)$ and $(R_3 \leftarrow R_3 - 3R_1)$ we get:

$\left\| \begin{matrix} 1 & -1 & -1 & 1 & | & 1 \\ 0 & 3 & 4 & -1 & | & -3 \\ 0 & 3 & 4 & -1 & | & -3 \end{matrix} \right\|$ so the third equation is unuseful and we solve the system;

$$\begin{cases} x_1 - x_2 = x_3 - x_4 + 1 \\ 3x_2 = -4x_3 + x_4 - 3 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 + x_3 - x_4 + 1 \\ x_2 = -\frac{4}{3}x_3 + \frac{1}{3}x_4 - 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 = -\frac{4}{3}x_3 + \frac{1}{3}x_4 - 1 + x_3 - x_4 + 1 \\ x_2 = -\frac{4}{3}x_3 + \frac{1}{3}x_4 - 1 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{1}{3}x_3 - \frac{2}{3}x_4 \\ x_2 = -\frac{4}{3}x_3 + \frac{1}{3}x_4 - 1 \end{cases}$$

which is the solution of our system having ∞^2 solutions.

I M 4) Firstly we find a vector $\mathbb{X}_2$ orthogonal to $\mathbb{X}_1$. If $\mathbb{X}_2 = (x, y, z)$ we put $\mathbb{X}_1 \cdot \mathbb{X}_2 = 0$ to get: $(1, 0, -1) \cdot (x, y, z) = x - z = 0 \Rightarrow z = x$. From $\mathbb{X}_2 = (x, y, x)$ we choose $x = 1$ and $y = 1$ to get $\mathbb{X}_2 = (1, 1, 1)$. To find the last vector $\mathbb{X}_3$ we start from (x, y, x) to find the relation for which $\mathbb{X}_3$ is orthogonal to $\mathbb{X}_2$, putting $(x, y, x) \cdot (1, 1, 1) = x + y + x = 0$ to get $y = -2x$. So $\mathbb{X}_3 = (x, -2x, x) \Rightarrow \mathbb{X}_3 = (1, -2, 1)$.

The three orthogonal vectors are: $\mathbb{X}_1 = (1, 0, -1); \mathbb{X}_2 = (1, 1, 1); \mathbb{X}_3 = (1, -2, 1)$.

To get an orthonormal basis for $\mathbb{R}^3$ we need three orthogonal unit vectors and so:

Since $\|\mathbb{X}_1\| = \sqrt{2}$ we get $\mathbb{V}_1 = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$;

Since $\|\mathbb{X}_2\| = \sqrt{3}$ we get $\mathbb{V}_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$;

Since $\|\mathbb{X}_3\| = \sqrt{6}$ we get $\mathbb{V}_1 = \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$.

The orthonormal basis for $\mathbb{R}^3$ is $\left\{\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right); \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right); \left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)\right\}$.

II M 1) From the equation $f(x, y) = x\,e^{x+y} + y\,e^{x-y} = 0$ we get $f(0, 0) = 0 + 0 = 0$ and so the point $(0, 0)$ satisfies the equation. Then :

$\nabla f(x, y) = (e^{x+y} + x\,e^{x+y} + y\,e^{x-y}; x\,e^{x+y} + e^{x-y} - y\,e^{x-y}) =$
$= ((1 + x)\,e^{x+y} + y\,e^{x-y}; x\,e^{x+y} + (1 - y)\,e^{x-y}) \Rightarrow \nabla f(0, 0) = (1, 1)$.

Since $f'_y(0, 0) \neq 0$ it is possible to define an implicit function $x \to y(x)$ whose derivative is:
$\frac{d\,y}{d\,x}(0) = -\frac{1}{1} = -1$.

II M 2) To solve the problem: $\begin{cases} \text{Max/min } f(x, y) = xy^2 - x^2 \\ \text{u.c. } x^2 + y^2 \leq 1 \end{cases}$, we observe that the objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a compact set, and so maximum and minimum values surely exist. The constraint is a circumference and so it is qualified.

The Lagrangian function is: $\Lambda(x, y, \lambda) = xy^2 - x^2 - \lambda(x^2 + y^2 - 1)$.

1) case $\lambda = 0$:

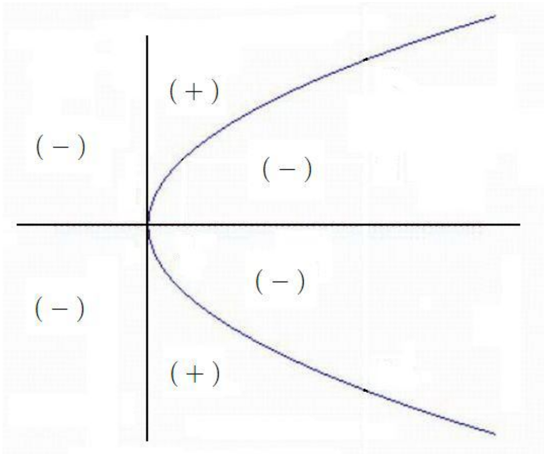
$$\begin{cases} \Lambda'_x = y^2 - 2x = 0 \\ \Lambda'_y = 2xy = 0 \\ x^2 + y^2 \le 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 + 0 \le 1 \end{cases} . \ \mathbb{H}(x,y) = \begin{Vmatrix} -2 & 2y \\ 2y & 2x \end{Vmatrix} \Rightarrow \mathbb{H}(0,0) = \begin{Vmatrix} -2 & 0 \\ 0 & 0 \end{Vmatrix}.$$

Surely $(0,0)$ is not a minimum point (since $-2 < 0$).

Studying the sign of the function in a neighborhood of $(0,0)$ we get: $f(x,y) = 0$ and

$f(x,y) = xy^2 - x^2 = x\left(y^2 - x\right) \Rightarrow f(x,y) \ge 0$ for $\begin{cases} x \ge 0 \\ y^2 \ge x \end{cases}$ or $\begin{cases} x \le 0 \\ y^2 \le x \end{cases}$ (impossible) .

As we see in the next figure, in every neighborhood of $(0,0)$ we have both negative and positive values and so the point $(0,0)$ is a saddle point.



2) case $\lambda \ne 0$:

$$\begin{cases} \Lambda'_x = y^2 - 2x - 2\lambda x = 0 \\ \Lambda'_y = 2xy - 2\lambda y = 2y(x - \lambda) = 0 \\ x^2 + y^2 = 1 \end{cases}$$ from which we have two possible systems.

I) $\begin{cases} 2x + 2\lambda x = 2x(1+\lambda) = 0 \\ y = 0 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 + 0 = 1 : \text{impossible} \end{cases}$ and

$\begin{cases} 1 + \lambda = 0 \\ y = 0 \\ x^2 = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda = -1 \end{cases}$ and $\begin{cases} x = -1 \\ y = 0 \\ \lambda = -1 \end{cases}$: since $\lambda < 0$ these two points may be minimum points;

II) $\begin{cases} y^2 - 2\lambda - 2\lambda^2 = 0 \\ x = \lambda \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} x = \lambda \\ y^2 = 2\lambda + 2\lambda^2 \\ \lambda^2 + 2\lambda + 2\lambda^2 = 1 \end{cases} \Rightarrow \begin{cases} x = \lambda \\ y^2 = 2\lambda + 2\lambda^2 \\ 3\lambda^2 + 2\lambda - 1 = 0 \end{cases} \Rightarrow$

$$3\lambda^2 + 2\lambda - 1 = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1+3}}{3} = \frac{-1 \pm 2}{3} \Rightarrow \lambda_1 = -1, \lambda_2 = \frac{1}{3}.$$

So we get three other solutions:

$\begin{cases} x = -1 \\ y^2 = 0 \\ \lambda = -1 \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = 0 \\ \lambda = -1 \end{cases}$ already seen, it may be a minimum point;

$\begin{cases} x = \frac{1}{3} \\ y^2 = \frac{2}{3} + \frac{2}{9} = \frac{8}{9} \\ \lambda = \frac{1}{3} \end{cases} \Rightarrow \begin{cases} x = \frac{1}{3} \\ y = \pm\frac{2\sqrt{2}}{3} \\ \lambda = \frac{1}{3} \end{cases}$ since $\lambda > 0$ these two points may be maximum points.

Since $f(1,0) = f(-1,0) = -1$ we see that $(1,0)$ and $(-1,0)$ are minumum points;

since $f\left(\frac{1}{3}, \frac{2\sqrt{2}}{3}\right) = f\left(\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right) = \frac{5}{27}$ we see that $\left(\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$ and $\left(\frac{1}{3}, -\frac{2\sqrt{2}}{3}\right)$ are maximum points.

II M 3) Since the function $f(x,y) = x\,e^{y-x} - y\,e^{x-y}$ is clearly a differentiable function in $\mathbb{R}^2$, we simply calculate $D_v f(x,y) = \nabla f(x,y) \cdot v$.
$\nabla f(x,y) = (e^{y-x} - x\,e^{y-x} - y\,e^{x-y}; x\,e^{y-x} - e^{x-y} + y\,e^{x-y}) =$
$= ((1-x)\,e^{y-x} - y\,e^{x-y}; x\,e^{y-x} + (y-1)\,e^{x-y})$.
So $\nabla f(0,0) = (1, -1)$ and $\nabla f(1,1) = (-1, 1)$. From this we get:
$D_v f(0,0) = \nabla f(0,0) \cdot v = (1, -1) \cdot (\cos\alpha, \sin\alpha) = \cos\alpha - \sin\alpha$
$D_v f(1,1) = \nabla f(1,1) \cdot v = (-1, 1) \cdot (\cos\alpha, \sin\alpha) = \sin\alpha - \cos\alpha$.
From $D_v f(0,0) = D_v f(1,1)$ we get $\cos\alpha - \sin\alpha = \sin\alpha - \cos\alpha \Rightarrow \sin\alpha = \cos\alpha$ which is satisfied for $\alpha = \frac{\pi}{4}$ and $\alpha = \frac{5\pi}{4}$.

II M 4) To analyze the nature of the stationary points of the function we apply first and second order conditions. For the first order conditions we pose:

$$\nabla f(x,y,z) = \mathbb{O} \Rightarrow \begin{cases} f'_x = y^2 - 2x = 0 \\ f'_y = 2xy - 2y = 2y\,(x-1) = 0 \\ f'_z = -2z = 0 \end{cases}$$ from which we get two systems:

I) $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$ and II) $\begin{cases} x = 1 \\ y^2 = 2 \\ z = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = \sqrt{2} \\ z = 0 \end{cases}$ and $\Rightarrow \begin{cases} x = 1 \\ y = -\sqrt{2} \\ z = 0 \end{cases}$.

For the second order conditions we construct the Hessian matrix:

$$\mathbb{H} = \left\| \begin{matrix} -2 & 2y & 0 \\ 2y & 2x-2 & 0 \\ 0 & 0 & -2 \end{matrix} \right\|$$. So we get:

$$\mathbb{H}(0,0,0) = \left\| \begin{matrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{matrix} \right\| \Rightarrow \begin{cases} |\mathbb{H}_1| = -2 < 0 \\ |\mathbb{H}_2| = 4 > 0 \\ |\mathbb{H}_3| = -8 < 0 \end{cases}$$ so $(0,0,0)$ is a maximum point;

$$\mathbb{H}\left(1, \sqrt{2}, 0\right) = \left\| \begin{matrix} -2 & 2\sqrt{2} & 0 \\ 2\sqrt{2} & 0 & 0 \\ 0 & 0 & -2 \end{matrix} \right\| \Rightarrow |\mathbb{H}_2| = \begin{vmatrix} -2 & 2\sqrt{2} \\ 2\sqrt{2} & 0 \end{vmatrix} = -8 < 0$$

$$\mathbb{H}\left(1, -\sqrt{2}, 0\right) = \left\| \begin{matrix} -2 & -2\sqrt{2} & 0 \\ -2\sqrt{2} & 0 & 0 \\ 0 & 0 & -2 \end{matrix} \right\| \Rightarrow |\mathbb{H}_2| = \begin{vmatrix} -2 & -2\sqrt{2} \\ -2\sqrt{2} & 0 \end{vmatrix} = -8 < 0$$

and so $\left(1, \sqrt{2}, 0\right)$ and $\left(1, -\sqrt{2}, 0\right)$ are saddle points.