Compito di Analisi Matematica del 23/3/2019 $\boxed{CAM1}$

IM1) Se $z=(1-i)^3 \Rightarrow 1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}\right)=\sqrt{2}\cdot\left(\cos\frac{7}{4}\pi+i\,\text{sen}\,\frac{7}{4}\pi\right)$;

$(1-i)^3=2\sqrt{2}\left(\cos\frac{21}{4}\pi+i\,\text{sen}\,\frac{21}{4}\pi\right)=2\sqrt{2}\cdot\left(\cos\frac{5}{4}\pi+i\,\text{sen}\,\frac{5}{4}\pi\right)$;

$\sqrt{(1-i)^3}=\sqrt{2\sqrt{2}}\cdot\left(\cos\left(\frac{5}{8}\pi+k\cdot\frac{2\pi}{2}\right)+i\,\text{sen}\left(\frac{5}{8}\pi+k\cdot\frac{2\pi}{2}\right)\right)$; $0\leq k\leq 1$.

Per $k=0$: $\sqrt{2\sqrt{2}}\cdot\left(\cos\frac{5}{8}\pi+i\,\text{sen}\,\frac{5}{8}\pi\right)$; per $k=1$: $\sqrt{2\sqrt{2}}\cdot\left(\cos\frac{13}{8}\pi+i\,\text{sen}\,\frac{13}{8}\pi\right)$.

IM2) $f(x;y)=\begin{cases}\dfrac{x^\alpha y}{x^2+y^2} & : (x;y)\neq(0;0)\\ 0 & : (x;y)=(0;0)\end{cases}$. Per verificare la continuità:

$\lim_{(x;y)\to(0;0)}\frac{x^\alpha y}{x^2+y^2}\Rightarrow\lim_{\rho\to0}\frac{\rho^{\alpha+1}\cdot\cos^\alpha\vartheta\cdot\text{sen}\vartheta}{\rho^2}=\lim_{\rho\to0}\rho^{\alpha-1}\cdot\cos^\alpha\vartheta\,\text{sen}\vartheta=0$ se $\alpha-1>0\Rightarrow$

$\Rightarrow\alpha>1$. Convergente uniforme in quanto $|\cos^\alpha\vartheta\cdot\text{sen}\vartheta|<1$.

$\frac{\partial f}{\partial x}(0;0)=\lim_{h\to0}\frac{h^\alpha\cdot0}{h^2}\cdot\frac{1}{h}=\lim_{h\to0}0=0=\frac{\partial f}{\partial y}(0;0)=\lim_{h\to0}\frac{0\cdot h}{h^2}\cdot\frac{1}{h}$. Quindi $\nabla f(0;0)=(0;0)$.

Per la differenziabilità: $\lim_{(x;y)\to(0;0)}\left(\frac{x^\alpha y}{x^2+y^2}-0-(0;0)(x-0;y-0)\right)\cdot\frac{1}{\sqrt{x^2+y^2}}\Rightarrow$

$\Rightarrow\lim_{\rho\to0}\frac{\rho^{\alpha+1}\cos^\alpha\vartheta\cdot\text{sen}\vartheta}{\rho^2}\cdot\frac{1}{\rho}=\lim_{\rho\to0}\rho^{\alpha-2}\cdot\cos^\alpha\vartheta\cdot\text{sen}\vartheta=0$ se $\alpha-2>0\Rightarrow\alpha>2$.

La funzione è differenziabile per $\alpha>2$.

IM3) $\begin{cases}f(x;y;z)=xe^y+ye^z-2exz=0\\ g(x;y;z)=\log xy-\log yz+\log xz=0\end{cases}\Rightarrow\begin{cases}f(1;1;1)=e+e-2e=0\\ g(1;1;1)=0-0+0=0\end{cases}$.

$\frac{\partial(f;g)}{\partial(x;y;z)}=\left\|\begin{matrix}e^y-2ez & xe^y+e^z & ye^z-2ex\\ \frac{1}{x}+\frac{1}{x} & \frac{1}{y}-\frac{1}{y} & -\frac{1}{z}+\frac{1}{z}\end{matrix}\right\|$; $\frac{\partial(f;g)}{\partial(x;y;z)}(1;1;1)=\begin{array}{c}\left\|\begin{matrix}-e & 2e & -e\\ 2 & 0 & 0\end{matrix}\right\|\\ \begin{matrix}(x) & (y) & (z)\end{matrix}\end{array}$.

Essendo $\begin{vmatrix}-e & 2e\\ 2 & 0\end{vmatrix}=-4e\neq0$ si può definire: $z\to(x(z);y(z))$.

$\frac{dx}{dz}=-\frac{\begin{vmatrix}-e & 2e\\ 0 & 0\end{vmatrix}}{\begin{vmatrix}-e & 2e\\ 2 & 0\end{vmatrix}}=-\frac{0}{-4e}=0$; $\frac{dy}{dz}=-\frac{\begin{vmatrix}-e & -e\\ 2 & 0\end{vmatrix}}{\begin{vmatrix}-e & 2e\\ 2 & 0\end{vmatrix}}=-\frac{2e}{-4e}=\frac{1}{2}$.

IM4) $f(x;y)=x^2y-xy^2$. Funzione differenziabile due volte $\forall (x;y)\in\mathbb{R}^2$.

Quindi $\mathcal{D}^2_{v,w} f(1;1)= v\cdot H(1;1)\cdot w^T$. $\nabla f(x;y)=(2xy-y^2;\ x^2-2xy)$.

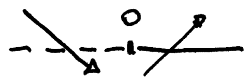
$H(x;y)=\begin{Vmatrix} 2y & 2x-2y \\ 2x-2y & -2x \end{Vmatrix} \Rightarrow H(1;1)=\begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix}$. versore di $(1;1)$: $\left(\frac{1}{\sqrt2};\frac{1}{\sqrt2}\right)$.

$$\mathcal{D}^2_{v,w} f(1;1)=\begin{Vmatrix}\cos\alpha & \operatorname{sen}\alpha\end{Vmatrix}\cdot\begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix}\cdot\begin{Vmatrix} \frac{1}{\sqrt2} \\ \frac{1}{\sqrt2} \end{Vmatrix}=\begin{Vmatrix}\cos\alpha & \operatorname{sen}\alpha\end{Vmatrix}\cdot\begin{Vmatrix} \sqrt2 \\ -\sqrt2 \end{Vmatrix}=\sqrt2(\cos\alpha-\operatorname{sen}\alpha)=0$$

$\Rightarrow \cos\alpha=\operatorname{sen}\alpha \Rightarrow \alpha=\frac{\pi}{4}$ o $\alpha=\frac{5}{4}\pi$.

IIM1) $\begin{cases} \operatorname{Max}(\min) f(x;y;z)=x^2+y^2+z^2-y+z \\ \text{s.v.} \begin{cases} x-y+z=1 \\ x+y-z=1 \end{cases} \end{cases} \Rightarrow \begin{cases} 2x=2 \\ z=x+y-1 \end{cases} \Rightarrow \begin{cases} x=1 \\ z=y \end{cases}$.

Studiamo quindi $f(z)=1+z^2+z^2-z+z=2z^2+1$. $f'(z)=4z\geq 0$ per $z\geq 0$.



Per $z=0$ punto di minimo. Da $z=0 \Rightarrow x=1$ e $y=0$.

Quindi $(1;0;0)$ è un punto di minimo per il problema.

Risolvendo il problema con la f. Lagrangiana avremo:

$\Lambda(x;y;z)=x^2+y^2+z^2-y+z-\lambda_1(x-y+z-1)-\lambda_2(x+y-z-1)$.

Condizioni del I Ordine: $\nabla\Lambda=\underline{0} \Rightarrow \begin{cases} \Lambda'_x=2x-\lambda_1-\lambda_2=0 \\ \Lambda'_y=2y-1+\lambda_1-\lambda_2=0 \\ \Lambda'_z=2z+1-\lambda_1+\lambda_2=0 \\ x-y+z=1 \\ x+y-z=1 \end{cases}$ che ha la soluzione $(1;0;0)$.

Condizioni del II Ordine:

Matrice Hessiana orlata:

$$\bar{H}(x;y;z;\lambda_1;\lambda_2)=\begin{Vmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 2 \end{Vmatrix}$$

. Essendo:

$$|\bar{H}_5|=\begin{vmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 2 \end{vmatrix}=16>0$$

ritroviamo che $(1;0;0)$ è un punto di minimo.

$\boxed{CAM3}$

II M2) $\begin{cases} x' = x - y - 1 \\ y' = x + y + 1 \end{cases} \Rightarrow \begin{cases} x' - x + y = -1 \\ -x + y' - y = 1 \end{cases} \Rightarrow \begin{vmatrix} D-1 & 1 \\ -1 & D-1 \end{vmatrix} \cdot (x) = \begin{vmatrix} -1 & 1 \\ 1 & D-1 \end{vmatrix} \Rightarrow$

$\Rightarrow \left((D-1)^2+1\right)(x) = 0 + 1 - 1 \Rightarrow (D^2 - 2D + 2)(x) = 0 \Rightarrow x'' - 2x' + 2x = 0.$

$\lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = 1 \pm \sqrt{1-2} = 1 \pm i$. $x(t) = c_1 e^t \operatorname{sen} t + c_2 e^t \cos t.$

$y(t) = x - x' - 1 = c_1 \cancel{e^t \operatorname{sen} t} + c_2 \cancel{e^t \cos t} - c_1 \cancel{e^t \operatorname{sen} t} - c_1 e^t \cos t - c_2 \cancel{e^t \cos t} + c_2 e^t \operatorname{sen} t - 1 \Rightarrow$

$y(t) = -c_1 e^t \cos t + c_2 e^t \operatorname{sen} t - 1.$

II M3) $y'' - y = e^x \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$. Soluzione omogenea: $y = c_1 e^x + c_2 e^{-x}$.

Ipotizziamo $y_0(x) = a \cdot x e^x \Rightarrow y_0' = a e^x + a x e^x$; $y_0'' = a e^x + a e^x + a x e^x \Rightarrow$

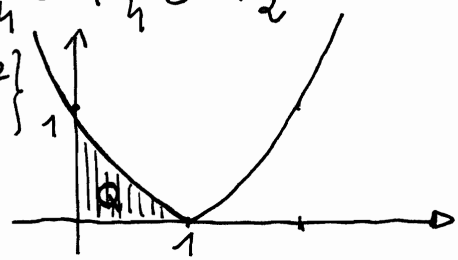
$y'' - y = 2a e^x + \cancel{a x e^x} - \cancel{a x e^x} = e^x \Rightarrow 2a e^x = e^x \Rightarrow a = \frac{1}{2}.$

Soluzione non omogenea: $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x$.

$y(0) = c_1 + c_2 = 1$; $y' = c_1 e^x - c_2 e^{-x} + \frac{1}{2} e^x + \frac{1}{2} x e^x \Rightarrow y'(0) = c_1 - c_2 + \frac{1}{2} \Rightarrow$

$\begin{cases} c_1 + c_2 = 1 \\ c_1 - c_2 = -\frac{1}{2} \end{cases} \Rightarrow \begin{cases} 2c_1 = \frac{1}{2} \\ c_1 + c_2 = 1 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{1}{4} \\ c_2 = \frac{3}{4} \end{cases}$. Soluzione: $y = \frac{1}{4} e^x + \frac{3}{4} e^{-x} + \frac{1}{2} x e^x$.

II M4) $\iint_Q 2xy \, dx\, dy$ con $Q = \{(x;y) : 0 \le x \le 1 ; 0 \le y \le (x-1)^2\}$



$\Rightarrow \int_0^1 \int_0^{(x-1)^2} (2y \cdot x \, dy)\, dx = \int_0^1 y^2 \Big|_0^{(x-1)^2} \cdot x \, dx =$

$= \int_0^1 x \cdot (x-1)^4 - 0 \, dx$. Per calcolare $\int x(x-1)^4 dx$ integriamo per parti:

$\int x(x-1)^4 dx = \frac{1}{5} x (x-1)^5 - \int \frac{1}{5} \cdot 1 \cdot (x-1)^5 dx = \frac{1}{5} x (x-1)^5 - \frac{1}{30} (x-1)^6$ da cui:

$\int_0^1 x(x-1)^4 dx = \left(\frac{1}{5} x (x-1)^5 - \frac{1}{30}(x-1)^6\right|_0^1 = (0-0) - \left(0 - \frac{1}{30}\right) = \frac{1}{30}.$