INTERMEDIATE TEST MATHEMATICS for ECONOMIC APPLICATIONS 7/12/2018

I M 1) $z = e^{1+3\pi i} = e \cdot e^{3\pi i} = e\,(\cos 3\pi + i \sin 3\pi) = e\,(\cos 3\pi + i \sin 3\pi) =$
$= e\,(\cos \pi + i \sin \pi) = -\,e$. The two square roots of z are:
$\sqrt{z} = \sqrt{e\,(\cos \pi + i \sin \pi)} = \sqrt{e}\left(\cos\left(\frac{\pi}{2} + k\pi\right) + i \sin\left(\frac{\pi}{2} + k\pi\right)\right)$ with $k = 0, 1$.
The two roots are:
$z_1 = \sqrt{e}\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = \sqrt{e}\,i$ and
$z_2 = \sqrt{e}\left(\cos\frac{3}{2}\pi + i \sin\frac{3}{2}\pi\right) = -\sqrt{e}\,i = -\,z_1$.

I M 2) The characteristic polynomial of $\mathbb{A}$ is $p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| =$

$$= \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & k-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (k-\lambda)\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (k-\lambda)\left((1-\lambda)^2 - 1\right) =$$

$$= (k-\lambda)\left(\lambda^2 - 2\lambda\right) = \lambda(\lambda-2)(k-\lambda).$$

Matrix $\mathbb{A}$ has the eingevalues $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = k$, thus $\mathbb{A}$ admits multiple eigenvalues if and only if $k = 0$ or $k = 2$.

If $k = 0$, an eigenvector associated to the eigenvalue $\lambda = 0$ is a vector $\mathbb{X} = \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix}$ satisfying the condition $\|\mathbb{A} - 0\,\mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix}$

or, in system form:

$$\begin{cases} x_1 + x_3 = 0 \\ 0 = 0 \\ x_1 + x_3 = 0 \end{cases} \Rightarrow x_3 = -\,x_1\text{, and so } \mathbb{X} = \begin{Vmatrix} x_1 \\ x_2 \\ -\,x_1 \end{Vmatrix} \forall\, x_1, x_2.$$

If $k = 2$, an eigenvector associated to the eigenvalue $\lambda = 2$ is a vector $\mathbb{X} = \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix}$ satisfying the condition $\|\mathbb{A} - 2\,\mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix}$

or, in system form:

$$\begin{cases} -\,x_1 + x_3 = 0 \\ 0 = 0 \\ x_1 - x_3 = 0 \end{cases} \Rightarrow x_3 = x_1\text{, and so } \mathbb{X} = \begin{Vmatrix} x_1 \\ x_2 \\ x_1 \end{Vmatrix} \forall\, x_1, x_2.$$

I M 3) From $f(1,1,1) = (3, 6, -3, 9) \Rightarrow \mathbb{A} \cdot (1,1,1) = (3, 6, -3, 9)$ we get:

$$\begin{Vmatrix} 1 & x_1 & y_1 \\ 2 & x_2 & y_2 \\ -1 & x_3 & y_3 \\ 3 & x_4 & y_4 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1+x_1+y_1 \\ 2+x_2+y_2 \\ -1+x_3+y_3 \\ 3+x_4+y_4 \end{Vmatrix} = \begin{Vmatrix} 3 \\ 6 \\ -3 \\ 9 \end{Vmatrix}$$, so the system:

$$\begin{cases} 1+x_1+y_1 = 3 \\ 2+x_2+y_2 = 6 \\ -1+x_3+y_3 = -3 \\ 3+x_4+y_4 = 9 \end{cases} \Rightarrow \begin{cases} y_1 = 2-x_1 \\ y_2 = 4-x_2 \\ y_3 = -2-x_3 \\ y_4 = 6-x_4 \end{cases}.$$

From $f(1,-1,1) = (3,4,-7,5) \Rightarrow \mathbb{A} \cdot (1,-1,1) = (3,4,-7,5)$ we get:

$$\begin{Vmatrix} 1 & x_1 & y_1 \\ 2 & x_2 & y_2 \\ -1 & x_3 & y_3 \\ 3 & x_4 & y_4 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ -1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1-x_1+y_1 \\ 2-x_2+y_2 \\ -1-x_3+y_3 \\ 3-x_4+y_4 \end{Vmatrix} = \begin{Vmatrix} 3 \\ 4 \\ -7 \\ 5 \end{Vmatrix}$$. From

$$\begin{cases} y_1 = 2-x_1 \\ y_2 = 4-x_2 \\ y_3 = -2-x_3 \\ y_4 = 6-x_4 \end{cases} \text{ we get } \begin{cases} 3-2x_1 = 3 \\ 6-2x_2 = 4 \\ -3-2x_3 = -7 \\ 9-2x_4 = 5 \end{cases} \text{ and so } \begin{cases} x_1 = 0 \text{ and } y_1 = 2 \\ x_2 = 1 \text{ and } y_2 = 3 \\ x_3 = 2 \text{ and } y_3 = -4 \\ x_4 = 2 \text{ and } y_4 = 4 \end{cases}.$$

So we get $\mathbb{A} = \begin{Vmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \\ -1 & 2 & -4 \\ 3 & 2 & 4 \end{Vmatrix}$.

The dimension of the Image of the linear map is equal to the Rank k of the matrix $\mathbb{A}$ while the dimension of its Kernel is equal to $n-k$, the difference between the dimension of the domain of the linear map and the Rank of the matrix $\mathbb{A}$.

For the Rank of $\mathbb{A}$, note that $C_3 = 2C_1 - C_2$, thus the matrix has not full rank while the determinant of the sub-matrix $\begin{Vmatrix} 1 & 0 \\ 2 & 1 \end{Vmatrix}$ is different from zero.

So the Rank of $\mathbb{A}$ is 2, the dimension of the Image of f is 2 and the dimension of the Kernel of f is 1.

I M 4) The three vectors $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ are linearly dependent if and only if the determinant of the matrix $\mathbb{X} = \| \mathbb{X}_1 \quad \mathbb{X}_2 \quad \mathbb{X}_3 \|$ is equal to 0.

$$|\mathbb{X}| = \begin{vmatrix} 6 & 4 & 2 \\ 2 & 8 & k \\ 2 & 4 & -6 \end{vmatrix} = 6\begin{vmatrix} 8 & k \\ 4 & -6 \end{vmatrix} - 4\begin{vmatrix} 2 & k \\ 2 & -6 \end{vmatrix} + 2\begin{vmatrix} 2 & 8 \\ 2 & 4 \end{vmatrix} =$$

$= -288 - 24k + 48 + 8k - 16 = -256 - 16k$; thus the three vectors are linearly dependent if and only if $k = -16$. For $k = -16$ we get $\mathbb{X}_3 = (2,-16,-6)$.

From $\alpha\mathbb{X}_1 + \beta\mathbb{X}_2 + \gamma\mathbb{X}_3 = \mathbb{O}$ we get:

$$\begin{cases} 6\alpha + 4\beta + 2\gamma = 0 \\ 2\alpha + 8\beta - 16\gamma = 0 \\ 2\alpha + 4\beta - 6\gamma = 0 \end{cases} \Rightarrow \begin{cases} 3\alpha + 2\beta + \gamma = 0 \\ \alpha + 4\beta - 8\gamma = 0 \\ \alpha + 2\beta - 3\gamma = 0 \end{cases} \Rightarrow \begin{cases} \gamma = -3\alpha - 2\beta \\ 2\beta - 5\gamma = 0 \\ \alpha + 2\beta + 9\alpha + 6\beta = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \gamma = -3\alpha - 2\beta \\ 2\beta - 5\gamma = 0 \\ 5\alpha + 4\beta = 0 \end{cases} \Rightarrow \begin{cases} \frac{2}{5}\beta = \frac{12}{5}\beta - 2\beta = \frac{2}{5}\beta \\ \gamma = \frac{2}{5}\beta \\ \alpha = -\frac{4}{5}\beta \end{cases}$$. So, for $\beta = 5$ we get:

$(\alpha,\beta,\gamma) = (-4,5,2) \Rightarrow 4\mathbb{X}_1 - 5\mathbb{X}_2 - 2\mathbb{X}_3 = \mathbb{O}$.

I M 5) Matrices $\mathbb{A}$ and $\mathbb{B}$ are similar if : $\mathbb{A}\cdot\mathbb{P}=\mathbb{P}\cdot\mathbb{B}\Rightarrow\mathbb{B}=\mathbb{P}^{-1}\cdot\mathbb{A}\cdot\mathbb{P}$.

$$|\mathbb{P}|=1\,,\;\; \mathbb{P}^{-1}=\frac{1}{|\mathbb{P}|}\cdot(Adj(\mathbb{P}))^T=1\cdot\left\|\begin{matrix}1 & -1\\ -2 & 3\end{matrix}\right\|^T=\left\|\begin{matrix}1 & -2\\ -1 & 3\end{matrix}\right\|.$$

So $\mathbb{B}=\left\|\begin{matrix}1 & -2\\ -1 & 3\end{matrix}\right\|\cdot\left\|\begin{matrix}1 & 2\\ 2 & -1\end{matrix}\right\|\cdot\left\|\begin{matrix}3 & 2\\ 1 & 1\end{matrix}\right\|=\left\|\begin{matrix}1 & -2\\ -1 & 3\end{matrix}\right\|\cdot\left\|\begin{matrix}5 & 4\\ 5 & 3\end{matrix}\right\|=$

$$=\left\|\begin{matrix}-5 & -2\\ 10 & 5\end{matrix}\right\|.$$