TASK MATHEMATICS for ECONOMIC APPLICATIONS 23/03/2019

I M 1) If $z = (1-i)^3$, calculate $\sqrt{z}$.

From $1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$ we get:

$(1-i)^3 = 2\sqrt{2}\left(\cos\frac{21\pi}{4} + i\sin\frac{21\pi}{4}\right) = 2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ and so:

$\sqrt{z} = \sqrt{2\sqrt{2}}\left(\cos\left(\frac{5\pi}{8} + k\frac{2\pi}{2}\right) + i\sin\left(\frac{5\pi}{8} + k\frac{2\pi}{2}\right)\right), 0 \le k \le 1$.

For $k = 0$: $\sqrt{2\sqrt{2}}\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right) = -\sqrt{\sqrt{2}-1} + i\sqrt{\sqrt{2}+1}$,

for $k = 1$: $\sqrt{2\sqrt{2}}\left(\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}\right) = +\sqrt{\sqrt{2}-1} - i\sqrt{\sqrt{2}+1}$.

I M 2) The characteristic polynomial of $\mathbb{A} = \begin{Vmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{Vmatrix}$ is :

$p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)\left[(1-\lambda)^2 - 1\right] =$

$= (2-\lambda)\left(\lambda^2 - 2\lambda\right) = \lambda(2-\lambda)(\lambda-2) = 0$. So we get the three eigenvalues $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 2$.

To find an eigenvector associated to the eigenvalue $\lambda = 0$ we must solve the system:

$$\|\mathbb{A} - 0\,\mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} 2x_1 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 0 \\ x_3 = -x_2 \end{cases}$$

and so the eigenvectors associated to the eigenvalue $\lambda = 0$ are $(0, x, -x)$.

For $x = 1$ we get $(0, 1, -1)$ and the corresponding unit vector $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Corresponding to the eigenvalue $\lambda = 2$ we must find two orthogonal eigenvectors.
We solve the system:

$$\|\mathbb{A} - 2\,\mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} \forall x_1 \\ x_3 = x_2 \end{cases}.$$

For $x_1 = x_2 = 1$ we get $(1, 1, 1)$ and the corresponding unit vector $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

To find a second eigenvector associated to the eigenvalue $\lambda = 2$ and orthogonal to the eigenvector $(1, 1, 1)$ we pose $(1, 1, 1) \cdot (x_1, x_2, x_2) = x_1 + 2x_2 = 0 \Rightarrow x_1 = -2x_2$ from which we get $(-2, 1, 1)$ and the corresponding unit vector $\left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$.

So an orthogonal matrix which diagonalizes $\mathbb{A}$ is $\mathbb{U} = \begin{Vmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{Vmatrix}$.

I M 3) From Rouchè-Capelli Theorem, if $\text{Rank}(\mathbb{A}) = \text{Rank}(\mathbb{A}|\mathbb{Y}) = k$ the system has ∞^{n-k} solutions, where n is the number of the variables; in our problem $n = 4$. We study the Rank of the augmented matrix: $\|\mathbb{A}|\mathbb{Y}\| = \left\| \begin{matrix} 1 & 1 & 1 & 1 & | & 1 \\ 1 & 1 & 1 & 2 & | & 1 \\ 3 & 3 & 3 & h & | & m \end{matrix} \right\|$.

By elementary operations on the rows: $(R_2 \leftarrow R_2 - R_1)$ and $(R_3 \leftarrow R_3 - 3R_1)$ we get :

$\left\| \begin{matrix} 1 & 1 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & h-3 & | & m-3 \end{matrix} \right\|$. By $(R_3 \leftarrow R_3 - (h-3) \cdot R_2)$ we get:

$\left\| \begin{matrix} 1 & 1 & 1 & 1 & | & 1 \\ 0 & 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & m-3 \end{matrix} \right\|$. And so :

$-\ \forall\, h$ and $m = 3$: $\text{Rank}(\mathbb{A}) = \text{Rank}(\mathbb{A}|\mathbb{Y}) = 2$: the system has ∞^2 solutions;

$-\ \forall\, h$ if $m \neq 3$: $\text{Rank}(\mathbb{A}) = 2 < \text{Rank}(\mathbb{A}|\mathbb{Y}) = 3$: the system has no solutions.

I M 4) Since the vector $\mathbb{X}$ has coordinates $(1, -1)$ in the basis $\mathbb{V} = \{(2,1);(1,1)\}$ we get:

$\mathbb{X} = \left\| \begin{matrix} 2 & 1 \\ 1 & 1 \end{matrix} \right\| \cdot \left\| \begin{matrix} 1 \\ -1 \end{matrix} \right\| = \left\| \begin{matrix} 1 \\ 0 \end{matrix} \right\|$. In the basis $\mathbb{W} = \{(3,1);(2,1)\}$ we have:

$\left\| \begin{matrix} 1 \\ 0 \end{matrix} \right\| = \left\| \begin{matrix} 3 & 2 \\ 1 & 1 \end{matrix} \right\| \cdot \left\| \begin{matrix} x_1 \\ x_2 \end{matrix} \right\| \Rightarrow \begin{cases} 3x_1 + 2x_2 = 1 \\ x_1 + x_2 = 0 \end{cases} \Rightarrow \begin{cases} 3x_1 - 2x_1 = 1 \\ x_2 = -x_1 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = -1 \end{cases}.$

So the coordinates of the vector $\mathbb{X}$ in the basis $\mathbb{W} = \{(3,1);(2,1)\}$ are $(1, -1)$. Exactly the same.

II M 1) From the equation $f(x,y,z) = x\,e^y + y\,e^z - xz = 0$ we get:

$f(1,0,1) = 1 + 0 - 1 = 0$ and so the point $(1,0,1)$ satisfies the equation. Then :

$\nabla f(x,y,z) = (e^y - z; x\,e^y + e^z; y\,e^z - x) \Rightarrow \nabla f(1,0,1) = (0, 1+e, -1)$.

Since $f'_z(1,0,1) = -1 \neq 0$ it is possible to define an implicit function $(x,y) \to z(x,y)$ whose derivatives are: $\dfrac{\partial z}{\partial x}(1,0) = -\dfrac{0}{-1} = 0; \dfrac{\partial z}{\partial y}(1,0) = -\dfrac{1+e}{-1} = 1+e$.

II M 2) To solve the problem: $\begin{cases} \text{Max/min } f(x,y,z) = x^2 + y^2 + z^2 \\ \text{u.c.: } \begin{cases} x - y + z = 1 \\ x + y - z = 1 \end{cases} \end{cases}$, we observe that the objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is not a compact set, but from the equations of the constraints we can easily explicitly solve respect one variable:

$\begin{cases} x - y + z = 1 \\ x + y - z = 1 \end{cases} \Rightarrow \begin{cases} 2x = 2 \\ z = x + y - 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ z = y \end{cases}$. Substituting we get:

$f(x,y,z) = f(1,z,z) = F(z) = 1 + z^2 + z^2 = 1 + 2z^2$. Since $F'(z) = 4z$ simply we get $F'(z) = 4z \geq 0$ for $z \geq 0$. For $z \leq 0$ the function $F(z)$ is a decreasing function, for $z \geq 0$ the function $F(z)$ is an increasing function and so the point $z = 0$ is a minimum point.

For $z = 0$ we have also $x = 1$ and $y = 0$ and so the point $(1,0,0)$ is the unique solution of the problem and it is a minimum point.

If we want to solve the problem using the traditional Lagrangian function with first and second order conditions we have:

$\Lambda(x,y,z,\lambda_1,\lambda_2) = x^2 + y^2 + z^2 - \lambda_1\,(x - y + z - 1) - \lambda_2\,(x + y - z - 1)$.

First order conditions bring to the system:

$$\begin{cases} \Lambda'_x = 2x - \lambda_1 - \lambda_2 = 0 \\ \Lambda'_y = 2y + \lambda_1 - \lambda_2 = 0 \\ \Lambda'_z = 2z - \lambda_1 + \lambda_2 = 0 \\ x - y + z = 1 \\ x + y - z = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ z = 0 \\ \lambda_1 = 1 \\ \lambda_2 = 1 \end{cases}$$

. For the second order conditions we use the bordered Hessian matrix: $\overline{\mathbb{H}}_5 = \begin{Vmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 2 \end{Vmatrix}$. We must calculate only $|\overline{\mathbb{H}}_5|$.

Since $|\overline{\mathbb{H}}_5| = \begin{vmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 1 & -1 & 0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 2 & 0 \\ 0 & -2 & -2 & 0 & 2 \end{vmatrix} =$

$$= 1 \cdot \begin{vmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 2 & 2 & 2 & 0 \\ -2 & -2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 \\ -2 & -2 & 0 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & 2 & 2 \end{vmatrix} =$$

$= 2 \cdot \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 2 & 2 \end{vmatrix} = 2 \cdot (4 + 4) = 16 > 0$, constraints are two, an even number, and since $|\overline{\mathbb{H}}_5| > 0$ we find again that the point $(1, 0, 0)$ is the unique solution of the problem and it is a minimum point.

II M 3) Since the function $f(x, y) = x^2 y - xy^2 + y$ is clearly a differentiable function in $\mathbb{R}^2$, we simply calculate $D_v f(1, 1) = \nabla f(1, 1) \cdot v$ and $D_w f(1, 1) = \nabla f(1, 1) \cdot w$.
$\nabla f(x, y) = \left(2xy - y^2; x^2 - 2xy + 1\right) \Rightarrow \nabla f(1, 1) = (1, 0)$.
So $D_v f(1, 1) = (1, 0) \cdot (\cos\alpha, \sin\alpha) = \cos\alpha$ while $D_w f(1, 1) = (1, 0) \cdot (1, 0) = 1$ and from this we get $\cos\alpha = 1 \Rightarrow \alpha = 0$.

II M 4) To analyze the nature of the stationary points of the function we apply first and second order conditions. For the first order conditions we pose:

$$\nabla f(x, y) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 2(x - 2y) - 2 \cdot 2 \cdot (y - 2x) = 0 \\ f'_y = -2 \cdot 2 \cdot (x - 2y) + 2 \cdot (y - 2x) = 0 \end{cases} \Rightarrow \begin{cases} 10x - 8y = 0 \\ -8x + 10y = 0 \end{cases}$$

from which we get the unique solution $\begin{cases} x = 0 \\ y = 0 \end{cases}$.

For the second order conditions we construct the Hessian matrix:
$\mathbb{H}(x, y) = \mathbb{H}(0, 0) = \begin{Vmatrix} 10 & -8 \\ -8 & 10 \end{Vmatrix}$. Since $\begin{cases} |\mathbb{H}_1| = 10 > 0 \\ |\mathbb{H}_2| = 100 - 64 = 36 > 0 \end{cases}$ we see that the point $(0, 0)$ is a minimum point.