TASK MATHEMATICS for ECONOMIC APPLICATIONS 4/06/2019

I M 1) If $z = e^{1-3\pi i}$, calculate $\sqrt[3]{z}$.

From $z = e^{1-3\pi i} = e \cdot e^{-3\pi i} = e \cdot (\cos(-3\pi) + i \sin(-3\pi)) =$
$= e \cdot (\cos(-\pi) + i \sin(-\pi)) = e \cdot (\cos \pi + i \sin \pi) = -e$.

So $\sqrt[3]{z} = \sqrt[3]{-e} = \sqrt[3]{e} \cdot \left(\cos\left(\frac{\pi}{3} + k\frac{2\pi}{3}\right) + i \sin\left(\frac{\pi}{3} + k\frac{2\pi}{3}\right)\right), 0 \le k \le 2$.

For $k = 0$: $\sqrt[3]{e} \cdot \left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right) = \sqrt[3]{e} \cdot \left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)$,

for $k = 1$: $\sqrt[3]{e} \cdot (\cos \pi + i \sin \pi) = -\sqrt[3]{e}$,

for $k = 2$: $\sqrt[3]{e} \cdot \left(\cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3}\right) = \sqrt[3]{e} \cdot \left(\frac{1}{2} - i\,\frac{\sqrt{3}}{2}\right)$.

I M 2) The matrix $\mathbb{A} = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$ admits the eigenvector $(1, 1)$ corresponding to the eigenvalue $\lambda = 0$ and the eigenvector $(1, -1)$ corresponding to the eigenvalue $\lambda = 1$. Find the matrix $\mathbb{A}$.

The matrix $\mathbb{A}$ satisfies :

$$\mathbb{A} \cdot \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} = 0 \cdot \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} a_{11} + a_{12} = 0 \\ a_{21} + a_{22} = 0 \end{cases} \Rightarrow \begin{cases} a_{12} = -a_{11} \\ a_{21} = -a_{22} \end{cases} .$$

The matrix $\mathbb{A}$ satisfies :

$$\mathbb{A} \cdot \begin{Vmatrix} 1 \\ -1 \end{Vmatrix} = 1 \cdot \begin{Vmatrix} 1 \\ -1 \end{Vmatrix} = \begin{Vmatrix} 1 \\ -1 \end{Vmatrix} \Rightarrow \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ -1 \end{Vmatrix} = \begin{Vmatrix} 1 \\ -1 \end{Vmatrix} \Rightarrow$$

$$\Rightarrow \begin{cases} a_{11} - a_{12} = 1 \\ a_{21} - a_{22} = -1 \end{cases} \Rightarrow \begin{cases} 2a_{11} = 1 \\ 2a_{22} = 1 \end{cases} \Rightarrow \begin{cases} a_{11} = \frac{1}{2} \\ a_{22} = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} a_{12} = -\frac{1}{2} \\ a_{21} = -\frac{1}{2} \end{cases} .$$

So $\mathbb{A} = \begin{Vmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{Vmatrix}$.

I M 3) Check if the matrix $\mathbb{A} = \begin{Vmatrix} 3 & 2 & 1 \\ 1 & 4 & 4 \\ 1 & 2 & 5 \end{Vmatrix}$ is a diagonalizable one.

The characteristic polynomial of $\mathbb{A} = \begin{Vmatrix} 3 & 2 & 1 \\ 1 & 4 & 4 \\ 1 & 2 & 5 \end{Vmatrix}$ is :

$$p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 3-\lambda & 2 & 1 \\ 1 & 4-\lambda & 4 \\ 1 & 2 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & \lambda-4 \\ 1 & 4-\lambda & 4 \\ 1 & 2 & 5-\lambda \end{vmatrix} =$$

$$= (2-\lambda)\left(\lambda^2 - 9\lambda + 20 - 8\right) + (\lambda-4)(2-4+\lambda) =$$
$$= (2-\lambda)\left(\lambda^2 - 9\lambda + 12\right) + (\lambda-4)(\lambda-2) = (2-\lambda)\left(\lambda^2 - 10\lambda + 16\right) =$$
$$= (2-\lambda)(\lambda-8)(\lambda-2) = 0 \Rightarrow \lambda_1 = \lambda_2 = 2 \text{ and } \lambda_3 = 8 .$$

To check if the matrix is a diagonalizable one we have only to study the Rank of $\|\mathbb{A} - 2\,\mathbb{I}\|$ to find the geometric multiplicity of $\lambda = 2$.

Since $\|\mathbb{A}-2\mathbb{I}\| = \begin{Vmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{Vmatrix}$ we immediately get $\text{Rank}(\mathbb{A}-2\,\mathbb{I}) = 2$ and so:
$m_2^g = 3-2 = 1 < 2 = m_2^a$ and the matrix is not a diagonalizable one.

I M 4) Given the linear map $f:\mathbb{R}^3 \to \mathbb{R}^2, f(\mathbb{X}) = \mathbb{A}\cdot\mathbb{X}$ with $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{Vmatrix}$ find a basis for the Kernel of such map and then find all the vectors having $(1,1)$ as their image.

To find a basis for the Kernel of such map we must solve the system:

$$\mathbb{A}\cdot\mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x+y+z=0 \\ x+y+2z=0 \end{cases} \Rightarrow \begin{cases} y = -x \\ z=0 \end{cases}.$$

Every vector belonging to the Kernel of such map is a vector of the form $(x,-x,0)$.
We see also that $\text{Dim(Ker)} = 1$ and a basis is formed by the vector $(1,-1,0)$.
To find all the vectors having $(1,1)$ as their image we must solve the system:

$$\mathbb{A}\cdot\mathbb{X} = \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} \Rightarrow \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} \Rightarrow \begin{cases} x+y+z=1 \\ x+y+2z=1 \end{cases} \Rightarrow \begin{cases} y = 1-x \\ z=0 \end{cases}$$

and so we get all the vectors of the form $(x,1-x,0) \Rightarrow f(x,1-x,0) = (1,1)$.

II M 1) Given the function $f(x,y) = x^2 - k\,xy + y^2$, v the unit vector of $(1,1)$ and w the unit vector of $(1,2)$, find the value for the parameter k for which $D_v f(1,1) = D_w f(1,1)$ and then calculate $D^2_{v,w} f(1,1)$.

$f(x,y) = x^2 - k\,xy + y^2$ is a twice differentiable function $\forall\,(x,y) \in \mathbb{R}^2$. So:
$D_v f(1,1) = \nabla f(1,1)\cdot v$ and $D_w f(1,1) = \nabla f(1,1)\cdot w$.
Easily we see that $\nabla f(x,y) = (2x-ky\,;2y-kx) \Rightarrow \nabla f(1,1) = (2-k;2-k)$;
$v = \left(\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right); w = \left(\frac{1}{\sqrt{5}};\frac{2}{\sqrt{5}}\right)$ to get:

$$D_v f(1,1) = (2-k;2-k)\cdot\left(\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right) = D_w f(1,1) = (2-k;2-k)\cdot\left(\frac{1}{\sqrt{5}};\frac{2}{\sqrt{5}}\right)$$

from which we get: $(2-k)\cdot\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right) = (2-k)\cdot\left(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}\right)$ and this equality can be satisfied only if $(2-k)=0 \Rightarrow k=2$.
So $f(x,y) = x^2+2xy+y^2$ and $D^2_{v,w} f(1,1) = v\cdot\mathbb{H}(1,1)\cdot w^T$.
From $\nabla f(x,y) = (2x-2y\,;2y-2x)$ we get
$\mathbb{H}(x,y) = \begin{Vmatrix} 2 & -2 \\ -2 & 2 \end{Vmatrix} = \mathbb{H}(1,1)$ and so :

$$D^2_{v,w} f(1,1) = \begin{Vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} 2 & -2 \\ -2 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{Vmatrix} = \begin{Vmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{Vmatrix} \cdot \begin{Vmatrix} \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{Vmatrix} = 0.$$

II M 2) Given the system $\begin{cases} f(x,y,z) = x^2+y^2+z^2 - 3xyz = 0 \\ g(x,y,z) = e^{x-y} + e^{y-z} - 2\,e^{z-x} = 0 \end{cases}$, satisfied at the point $P=(1,1,1)$, verify that it is possible to define an implicit function $x \to (y(x),z(x))$ and then calculate the derivatives of such function at $x=1$.

From $\dfrac{\partial(f,g)}{\partial(x,y,z)} = \left\| \begin{matrix} 2x-3yz & 2y-3xz & 2z-3xy \\ e^{x-y}+2\,e^{z-x} & -\,e^{x-y}+e^{y-z} & -\,e^{y-z}-2\,e^{z-x} \end{matrix} \right\|$ we get:
$\dfrac{\partial(f,g)}{\partial(x,y,z)}(1,1,1) = \left\| \begin{matrix} -1 & -1 & -1 \\ 3 & 0 & -3 \end{matrix} \right\|$ and since $\begin{vmatrix} -1 & -1 \\ 0 & -3 \end{vmatrix} = 3 \neq 0$ it is possible to define an implicit function $x \to (y(x), z(x))$. For its derivatives we get:

$$\frac{dy}{dx} = -\frac{\begin{vmatrix} -1 & -1 \\ 3 & -3 \end{vmatrix}}{\begin{vmatrix} -1 & -1 \\ 0 & -3 \end{vmatrix}} = -\frac{6}{3} = -2; \quad \frac{dz}{dx} = -\frac{\begin{vmatrix} -1 & -1 \\ 0 & 3 \end{vmatrix}}{\begin{vmatrix} -1 & -1 \\ 0 & -3 \end{vmatrix}} = -\frac{-3}{3} = 1\,.$$

II M 3) Solve the problem: $\begin{cases} \text{Max/min } f(x,y) = x^2 + y \\ \text{u.c.: } \begin{cases} x^2+y^2 \leq 1 \\ y \geq 0 \end{cases} \end{cases}$.

The objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a compact set, and so surely exist maximum and minimum values.
The Lagrangian function is: $\Lambda(x,y,\lambda) = x^2 + y - \lambda_1\big(x^2+y^2-1\big) - \lambda_2(-y)$.

1) case $\lambda_1 = \lambda_2 = 0$:
$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = 1 \neq 0 \end{cases}$ and so we don't get any internal stationary point.

2) case $\lambda_1 \neq 0; \lambda_2 = 0$:
$\begin{cases} \Lambda'_x = 2x - 2\lambda_1 x = 2x(1-\lambda_1) = 0 \\ \Lambda'_y = 1 - 2\lambda_1 y = 0 \\ x^2+y^2 = 1 \\ y \geq 0 \end{cases}$ from which we get two systems:

s1) $\begin{cases} x = 0 \\ \lambda_1 = \frac{1}{2y} \\ y^2 = 1 \\ y \geq 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = \frac{1}{2} \\ 1 \geq 0 \end{cases}$ and since $\lambda_1 > 0$ this may be a maximum point;

s2) $\begin{cases} 1-\lambda_1 = 0 \\ y = \frac{1}{2\lambda_1} \\ x^2+y^2 = 1 \\ y \geq 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 1 \\ y = \frac{1}{2} \\ x^2 = \frac{3}{4} \\ \frac{1}{2} \geq 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 1 \\ y = \frac{1}{2} \\ x = \frac{\sqrt{3}}{2} \\ \frac{1}{2} \geq 0 \end{cases}$ and $\begin{cases} \lambda_1 = 1 \\ y = \frac{1}{2} \\ x = -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \geq 0 \end{cases}$ and since $\lambda_1 > 0$ both points may be maximum points.

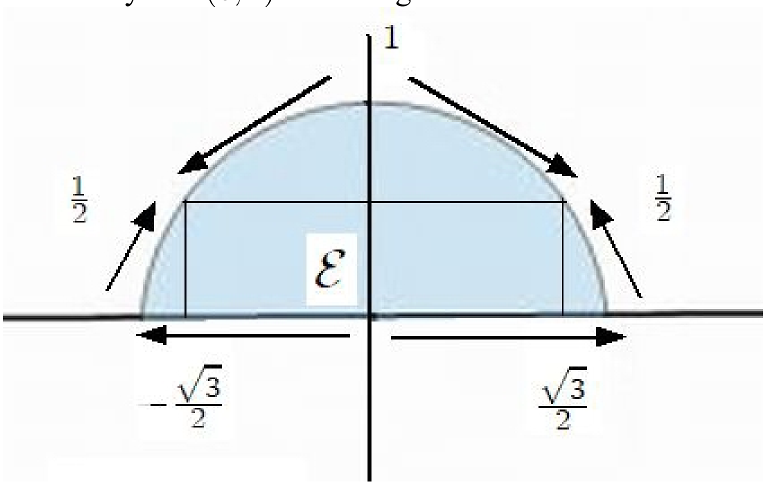
3) case $\lambda_1 = 0; \lambda_2 \neq 0$:
$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = 1 + \lambda_2 = 0 \\ y = 0 \\ x^2+y^2 \leq 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda_2 = -1 \\ 0+0 \leq 1 \end{cases}$ and since $\lambda_2 < 0$ this may be a minimum point.

4) case $\lambda_1 \neq 0; \lambda_2 \neq 0$:
$\begin{cases} \Lambda'_x = 2x - 2\lambda_1 x = 0 \\ \Lambda'_y = 1 - 2\lambda_1 y + \lambda_2 = 0 \\ \begin{cases} x^2+y^2 = 1 \\ y = 0 \end{cases} \end{cases} \Rightarrow \begin{cases} \Lambda'_x = 2x - 2\lambda_1 x = 0 \\ \Lambda'_y = 1 - 2\lambda_1 y + \lambda_2 = 0 \\ \begin{cases} x = 1 \\ y = 0 \end{cases} \text{ and } \begin{cases} x = -1 \\ y = 0 \end{cases} \end{cases}$ and we get two systems:

s1) $\begin{cases} 2-2\lambda_1=0 \\ 1+\lambda_2=0 \\ x=1 \\ y=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=0 \\ \lambda_1=1 \\ \lambda_2=-1 \end{cases}$: since $\lambda_1>0$ and $\lambda_2<0$ this point is not a maximum nor a minimum point;

s2) $\begin{cases} -2+2\lambda_1=0 \\ 1+\lambda_2=0 \\ x=-1 \\ y=0 \end{cases} \Rightarrow \begin{cases} x=-1 \\ y=0 \\ \lambda_1=1 \\ \lambda_2=-1 \end{cases}$: since $\lambda_1>0$ and $\lambda_2<0$ this point is not a maximum nor a minimum point.

Surely $(0,0)$ is the minimum point $(f(0,0)=0)$. If we study the objective function in the points satisfying the equation $x^2+y^2=1 \Rightarrow x^2=1-y^2$ substituting we get :
$f(y)=1-y^2+y$ from which we get $f'(y)=1-2y \geq 0$ for $y \leq \dfrac{1}{2}$.
So $\left(\dfrac{\sqrt{3}}{2};\dfrac{1}{2}\right)$ and $\left(-\dfrac{\sqrt{3}}{2};\dfrac{1}{2}\right)$ are maximum points, $\left(f\left(\frac{\sqrt{3}}{2};\frac{1}{2}\right)=f\left(-\frac{\sqrt{3}}{2};\frac{1}{2}\right)=1\right)$ while $(0;1)$ is nothing. If we study the bordered Hessian matrix for the equality constraint $x^2+y^2=1$ at point $(0;1)$ we get: $\overline{\mathbb{H}}(x,y,\lambda_1)=\left\|\begin{matrix} 0 & 2x & 2y \\ 2x & 2-2\lambda_1 & 0 \\ 2y & 0 & -2\lambda_1 \end{matrix}\right\|$ from which we get $\left|\overline{\mathbb{H}}\left(0,1,\dfrac{1}{2}\right)\right|=\begin{vmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{vmatrix}=2\cdot(-2)<0$ and so the point may be a minimum point. Another reason to say that $(0,1)$ is nothing.



II M 4) Given the function $f(x,y)=x^2-xy-x+y^2$ analyze the nature of its stationary points.

To analyze the nature of the stationary points of the function we apply first and second order conditions. For the first order conditions we pose:
$\nabla f(x,y)=\mathbb{O} \Rightarrow \begin{cases} f'_x=2x-y-1=0 \\ f'_y=-x+2y=0 \end{cases} \Rightarrow \begin{cases} 3y=1 \\ x=2y \end{cases}$ from which we get the unique solution $\begin{cases} x=\frac{2}{3} \\ y=\frac{1}{3} \end{cases}$.

For the second order conditions we construct the Hessian matrix:

$\mathbb{H}(x,y) = \mathbb{H}\left(\frac{2}{3}, \frac{1}{3}\right) = \left\| \begin{matrix} 2 & -1 \\ -1 & 2 \end{matrix} \right\|$. Since $\begin{cases} |\mathbb{H}_1| = 2 > 0 \\ |\mathbb{H}_2| = 4 - 1 = 3 > 0 \end{cases}$ we see that the point $\left(\frac{2}{3}, \frac{1}{3}\right)$ is a minimum point.