TASK MATHEMATICS for ECONOMIC APPLICATIONS 5/07/2019

I M 1) If $z_1 = e^{-1+3\pi i}$ and $z_2 = e\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, calculate $\sqrt{z_1 \cdot z_2}$.

From $z_1 = e^{-1+3\pi i} = e^{-1} \cdot e^{3\pi i} = e^{-1}(\cos 3\pi + i\sin 3\pi) =$
$= e^{-1}(\cos\pi + i\sin\pi) = -e^{-1}$.
So $z_1 \cdot z_2 = e^{-1} \cdot (\cos\pi + i\sin\pi) \cdot e\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) =$
$= e^{-1} \cdot e\left(\cos\left(\pi + \frac{3\pi}{4}\right) + i\sin\left(\pi + \frac{3\pi}{4}\right)\right) = 1 \cdot \left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$.
So $\sqrt{z_1 \cdot z_2} = \sqrt{1} \cdot \left(\cos\left(\frac{7\pi}{8} + k\frac{2\pi}{2}\right) + i\sin\left(\frac{7\pi}{8} + k\frac{2\pi}{2}\right)\right), 0 \le k \le 1$.
For $k = 0$: $\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}$, for $k = 1$: $\cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{Vmatrix}$ find an orthogonal matrix which diagonalizes $\mathbb{A}$.
The matrix is a symmetric one, and so it is surely diagonalizable by an orthogonal matrix.

From $|\mathbb{A} - \lambda\,\mathbb{I}| = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & -1 & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda+1 & 1 & 0 & 0 \\ 1-\lambda & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & -1 \\ 0 & 0 & -1 & -\lambda \end{vmatrix} =$

$= (1-\lambda)\begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{vmatrix} - (1-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 0 & -\lambda & -1 \\ 0 & -1 & -\lambda \end{vmatrix} =$
$= (1-\lambda)(-\lambda)(\lambda^2 - 1) - (1-\lambda)\,1\,(\lambda^2 - 1) = (\lambda - 1)(\lambda + 1)(\lambda^2 - 1) = 0$
for $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = \lambda_4 = -1$.
For finding the corresponding eigenvectors we solve two systems.

$\|\mathbb{A} - 1 \cdot \mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow$

$\Rightarrow \begin{cases} x_1 - x_2 = 0 \\ x_3 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = x_2 \\ x_3 = -x_4 \end{cases}$ and so the eigenvectors $(x_1, x_1, x_3, -x_3)$ from which we get two orthogonal eigenvectors $\mathbb{X}_1 = (1, 1, 0, 0)$ and $\mathbb{X}_2 = (0, 0, 1, -1)$.

$\|\mathbb{A} - (-1) \cdot \mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow$

$\Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_3 - x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -x_2 \\ x_3 = x_4 \end{cases}$ and so the eigenvectors $(x_1, -x_1, x_3, x_3)$ from which we get two orthogonal eigenvectors $\mathbb{X}_3 = (1, -1, 0, 0)$ and $\mathbb{X}_4 = (0, 0, 1, 1)$.

Using the corresponding unit vectors as columns we find the orthogonal matrix:

$$\mathbb{U}=\begin{Vmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{Vmatrix}.$$

I M 3) The matrix $\mathbb{A}=\begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$ admits the eigenvectors $(1,1)$ and $(1,-1)$ both corresponding to the eigenvalue $\lambda=1$. Find the matrix $\mathbb{A}$.

The matrix $\mathbb{A}$ satisfies :

$$\mathbb{A}\cdot\begin{Vmatrix} 1 \\ 1 \end{Vmatrix}=1\cdot\begin{Vmatrix} 1 \\ 1 \end{Vmatrix}=\begin{Vmatrix} 1 \\ 1 \end{Vmatrix}\Rightarrow\begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}\cdot\begin{Vmatrix} 1 \\ 1 \end{Vmatrix}=\begin{Vmatrix} 1 \\ 1 \end{Vmatrix}\Rightarrow$$

$$\Rightarrow\begin{cases} a_{11}+a_{12}=1 \\ a_{21}+a_{22}=1 \end{cases}\Rightarrow\begin{cases} a_{12}=1-a_{11} \\ a_{21}=1-a_{22} \end{cases}.$$

The matrix $\mathbb{A}$ satisfies also :

$$\mathbb{A}\cdot\begin{Vmatrix} 1 \\ -1 \end{Vmatrix}=1\cdot\begin{Vmatrix} 1 \\ -1 \end{Vmatrix}=\begin{Vmatrix} 1 \\ -1 \end{Vmatrix}\Rightarrow\begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}\cdot\begin{Vmatrix} 1 \\ -1 \end{Vmatrix}=\begin{Vmatrix} 1 \\ -1 \end{Vmatrix}\Rightarrow$$

$$\Rightarrow\begin{cases} a_{11}-a_{12}=1 \\ a_{21}-a_{22}=-1 \end{cases}\Rightarrow\begin{cases} a_{11}-1+a_{11}=1 \\ 1-a_{22}-a_{22}=-1 \end{cases}\Rightarrow\begin{cases} 2a_{11}=2 \\ 2a_{22}=2 \end{cases}\Rightarrow\begin{cases} a_{11}=1 \\ a_{22}=1 \end{cases}.$$

So $\begin{cases} a_{12}=0 \\ a_{21}=0 \end{cases}$ and $\mathbb{A}=\begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$.

I M 4) Given the vector subspace $\mathbb{W}$ of $\mathbb{R}^3$ having base $\{\mathbb{W}_1=(1,0,1);\mathbb{W}_2=(1,1,0)\}$, check the values for the parameter k for which the vector $\mathbb{X}=(1,1,k)$ belongs to $\mathbb{W}$.

Our problem is like checking for the existence of solutions of a linear system having this augmented matrix: $(\mathbb{A}|\mathbb{Y})=\begin{Vmatrix} 1 & 1 & | & 1 \\ 0 & 1 & | & 1 \\ 1 & 0 & | & k \end{Vmatrix}$. To get $\text{Rank}(\mathbb{A})=\text{Rank}(\mathbb{A}|\mathbb{Y})$ so to satisfy Rouchè-Capelli Theorem, since $\text{Rank}(\mathbb{A})=2$, we simply need to get $\text{Det}(\mathbb{A}|\mathbb{Y})=0$ and so:

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & k \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & k-1 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & k \end{vmatrix}=k=0.$$

So the vector $\mathbb{X}=(1,1,k)$ belongs to $\mathbb{W}$ only if $k=0$.

II M 1) Given the function $f(x,y)=x\,e^y-y\,e^x$ and the unit vector $v=(\cos\alpha,\sin\alpha)$, find the values of α for which $D^2_{v,v}f(1,1)=e$.

$f(x,y)=x\,e^y-y\,e^x$ is a twice differentiable function $\forall\,(x,y)\in\mathbb{R}^2$. So:

$D_vf(1,1)=\nabla f(1,1)\cdot v$ and $D^2_{v,v}f(1,1)=v\cdot\mathbb{H}(1,1)\cdot v^T$.

We get $\nabla f(x,y)=(e^y-y\,e^x\,;x\,e^y-e^x)\Rightarrow\nabla f(1,1)=(0,0)$;

From $\nabla f(x,y)=(e^y-y\,e^x\,;x\,e^y-e^x)$ we get $\mathbb{H}(x,y)=\begin{Vmatrix} -y\,e^x & e^y-e^x \\ e^y-e^x & x\,e^y \end{Vmatrix}$ and so:

$$\mathbb{H}(1,1)=\begin{Vmatrix} -e & 0 \\ 0 & e \end{Vmatrix}\Rightarrow D^2_{v,v}f(1,1)=\|\cos\alpha\quad\sin\alpha\|\cdot\mathbb{H}(1,1)\cdot\begin{Vmatrix} \cos\alpha \\ \sin\alpha \end{Vmatrix}=$$

$$=\|\cos\alpha\quad\sin\alpha\|\cdot\begin{Vmatrix} -e & 0 \\ 0 & e \end{Vmatrix}\cdot\begin{Vmatrix} \cos\alpha \\ \sin\alpha \end{Vmatrix}=\|\cos\alpha\quad\sin\alpha\|\cdot\begin{Vmatrix} -e\cos\alpha \\ e\sin\alpha \end{Vmatrix}=$$

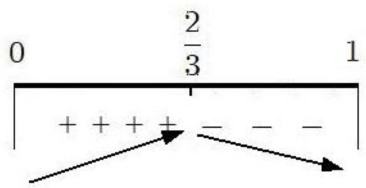
$$=-e\cos^2\alpha+e\sin^2\alpha=-e\cos 2\alpha=e\text{ for :}$$

$\cos 2\alpha = -1 \Rightarrow 2\alpha = \pi \Rightarrow \alpha = \dfrac{\pi}{2}$ or $2\alpha = 3\pi \Rightarrow \alpha = \dfrac{3\pi}{2}$.

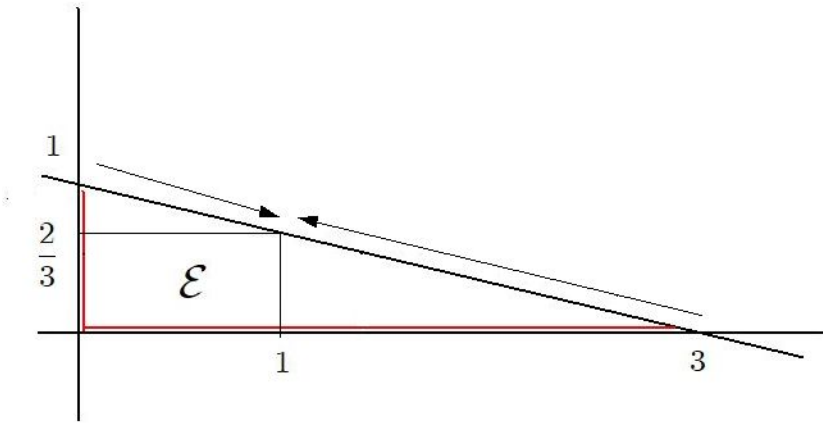
II M 2) Given the equation $f(x,y,z) = x\,e^{y-z} - y\,e^{x-z} = 0$, satisfied at $P = (0,0,0)$, verify that it is possible to define an implicit function $(x,z) \to y(x,z)$ and then calculate the derivatives of such function at $y = 0$.
From $\nabla f(x,y,z) = (e^{y-z} - y\,e^{x-z}\,; x\,e^{y-z} - e^{x-z};\, -x\,e^{y-z} + y\,e^{x-z})$ we get :
$\nabla f(0,0,0) = (1\,;-1;0)$ and since $f'_y(0,0,0) = -1 \neq 0$ it is possible to define an implicit function $(x,z) \to y(x,z)$. For its derivatives we have:
$\dfrac{\partial y}{\partial x} = -\dfrac{1}{-1} = 1;\ \dfrac{\partial y}{\partial z} = -\dfrac{0}{-1} = 0$.

II M 3) Solve the problem: $\begin{cases} \text{Max/min } f(x,y) = x\,y^2 \\ \text{u.c.: } \begin{cases} 3y + x - 3 \le 0 \\ 0 \le x \\ 0 \le y \end{cases} \end{cases}$.

The objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a triangle in the first quadrant of the real plan, i.e. a compact set, and so surely exist maximum and minimum values.
It is not convenient to use Kuhn-Tucker's conditions. We check for free maximum or minimum points and then we study the problem in the boundary points.
In the first quadrant, since $x \ge 0$ and $y \ge 0$, we get $f(x,y) = x\,y^2 \ge 0$, and so all the points belonging to the axes $x = 0$ and $y = 0$, where $f(x,0) = f(0,y) = 0$, are minimum points.
For free maximum or minimum points we get:
$\begin{cases} f'_x = y^2 = 0 \\ f'_y = 2xy = 0 \end{cases} \Rightarrow \begin{cases} \forall x \\ y = 0 \end{cases}$, points just studied.
It remains only to study the points satisfying $3y + x - 3 = 0 \Rightarrow x = 3 - 3y$.
Substituting we get: $f(3-3y, y) = (3-3y)\,y^2 = 3y^2 - 3y^3$ and deriving we get:
$f'(y) = 6y - 9y^2 = 3y\,(2-3y) \ge 0$ for $0 \le y \le \dfrac{2}{3}$.



So the point $\left(1;\dfrac{2}{3}\right)$ is the maximum point, with $f\left(1;\dfrac{2}{3}\right) = \dfrac{4}{9}$ while all the points $(x,0)$ and $(0,y)$ are minimum points (red lines)



II M 4) Given the function $f(x,y,z) = x^3 - 4x^2 + y^2 + z^2 + 5x - 2y$ analyze the nature of its stationary points.

To analyze the nature of the stationary points of the function we apply first and second order conditions. For the first order conditions we pose:

$\nabla f(x,y,z) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 3x^2 - 8x + 5 = 0 \\ f'_y = 2y - 2 = 0 \\ f'_z = 2z = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{5}{3} \text{ or } x = 1 \\ y = 1 \\ z = 0 \end{cases}$ and so we get two stationary points: $P_1 = \left(\frac{5}{3}; 1; 0\right)$ and $P_1 = (1;1;0)$. For the second order conditions we construct the Hessian matrix: $\mathbb{H}(x,y,z) = \begin{Vmatrix} 6x-8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{Vmatrix}$.

$\mathbb{H}\left(\frac{5}{3}; 1; 0\right) = \begin{Vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{Vmatrix}$. Since $\begin{cases} |\mathbb{H}_1| = 2 > 0 \\ |\mathbb{H}_2| = 4 > 0 \\ |\mathbb{H}_3| = 8 > 0 \end{cases}$ the point P_1 is a minimum point.

$\mathbb{H}(1;1;0) = \begin{Vmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{Vmatrix}$. Since $\begin{cases} |\mathbb{H}_1| = -2 < 0 \\ |\mathbb{H}_1| = 2 > 0 \end{cases}$ the point P_2 is a saddle point.