TASK MATHEMATICS for ECONOMIC APPLICATIONS 3/09/2019

I M 1) Determine all the roots of the equation $x^4 - 16 = 0$.
Two possible procedures for the solution.
From $x^4 - 16 = (x^2 - 4)(x^2 + 4) = 0$ we get:
$x^2 - 4 = 0 \Rightarrow x = \pm\sqrt{4} \Rightarrow x = \pm 2$ and
$x^2 + 4 = 0 \Rightarrow x = \pm\sqrt{-4} = \pm\sqrt{-1}\sqrt{4} \Rightarrow x = \pm 2i$. Or:
From $x^4 - 16 = 0$ we get $x = \sqrt[4]{16}$ and so, since $16 = 16 \cdot (\cos 0 + i \sin 0)$ we get:
$$\sqrt[4]{16} = \sqrt[4]{16} \cdot \left(\cos\left(0 + k\,\frac{2\pi}{4}\right) + i \sin\left(0 + k\,\frac{2\pi}{4}\right)\right) = 2 \cdot \left(\cos\left(k\,\frac{\pi}{2}\right) + i \sin\left(k\,\frac{\pi}{2}\right)\right)$$
for $0 \le k \le 3$ to get:
$k = 0 : 2 \cdot (\cos 0 + i \sin 0) = 2$; $k = 1 : 2 \cdot \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) = 2i$;
$k = 2 : 2 \cdot (\cos\pi + i \sin\pi) = -2$; $k = 3 : 2 \cdot \left(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}\right) = -2i$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & k \end{Vmatrix}$ determine the value of the parameter k for which the matrix admits the imaginary unit i as an eigenvalue. For this value of k, find all the eigenvalues of the matrix and check if it is diagonalizable or not.

From $|\mathbb{A} - \lambda\mathbb{I}| = 0$ we get $\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & k-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(k-\lambda) = 0$ to get the eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = k$. So the matrix admits the imaginary unit i as an eigenvalue if and only if $k = i$. So we get $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & i \end{Vmatrix}$. To check if the matrix is diagonalizable or not we have to study only the multiple eigenvalue $\lambda = 1$ and its geometric multiplicity. Since $\text{Rank}\,\|\mathbb{A} - 1 \cdot \mathbb{I}\| = \text{Rank}\begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & i-1 \end{Vmatrix} = 2$, we get $m_1^g = 3 - 2 = 1 < m_1^a = 2$ and so the matrix is not a diagonalizable one.

I M 3) Given the linear system $\begin{cases} x_1 + 2x_2 - x_3 + x_4 = 1 \\ x_1 + x_2 + m\,x_3 + x_4 = 2 \\ 2x_1 + 3x_2 + 2x_3 + kx_4 = 3 \end{cases}$, check for the existence and number of its solutions on varying the parameters m and k.

To apply Rouchè-Capelli theorem we study the Rank of the matrix and the Rank of the augmented matrix:
$$\begin{Vmatrix} 1 & 2 & -1 & 1 & | & 1 \\ 1 & 1 & m & 1 & | & 2 \\ 2 & 3 & 2 & k & | & 3 \end{Vmatrix} \to \begin{Vmatrix} 1 & 2 & -1 & 1 & | & 1 \\ 0 & -1 & m+1 & 0 & | & 1 \\ 0 & 0 & 3-m & k-2 & | & 0 \end{Vmatrix}$$
having used elementary operations: $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_2 - R_1$. So:
if $m \ne 3$ or $k \ne 2$: $\text{Rank}(\mathbb{A}) = \text{Rank}(\mathbb{A}|\mathbb{Y}) = 3$ and the system has $\infty^{4-3} = \infty^1$ solutions;
if $m = 3$ and $k = 2$: $\text{Rank}(\mathbb{A}) = \text{Rank}(\mathbb{A}|\mathbb{Y}) = 2$ and the system has $\infty^{4-2} = \infty^2$ solutions.

I M 4) Given the two orthogonal vectors $\mathbb{X}_1 = (1,1,0)$ and $\mathbb{X}_2 = (1,\,-1,0)$, find a third vector $\mathbb{X}_3$ orthogonal to $\mathbb{X}_1$ and $\mathbb{X}_2$, so as to create a basis for $\mathbb{R}^3$. Then find the coordinates of the vector $\mathbb{Y} = (1,1,1)$ in this basis.

If $\mathbb{X}_3 = (x,y,z)$, since we need $\mathbb{X}_1 \cdot \mathbb{X}_3 = \mathbb{X}_2 \cdot \mathbb{X}_3 = 0$ we get:

$\begin{cases} (1,1,0)\cdot(x,y,z) = x+y=0 \\ (1,\,-1,0)\cdot(x,y,z) = x-y=0 \end{cases}$ whose solution is $x=0, y=0, \forall\, z$. If we choose $z=1$ we get $\mathbb{X}_3 = (0,0,1)$ and so the basis is $\mathbb{X} = \{(1,1,0);(1,\,-1,0);(0,0,1)\}$.

To find the coordinates of the vector $\mathbb{Y} = (1,1,1)$ in this basis we must solve the system:

$$\left\|\begin{matrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix}\right\| = \left\|\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}\right\| \Rightarrow \begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 1 \\ x_3 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = 1 \end{cases}.$$

II M 1) Given $f(x,y) = x\,e^y - y\,e^x$, determine all the directions $v = (\cos\alpha, \sin\alpha)$ for which it results $\mathcal{D}_v f(0,0) = \mathcal{D}^2_{vv} f(0,0)$.

$f(x,y) = x\,e^y - y\,e^x$ is a twice differentiable function $\forall\,(x,y)\in\mathbb{R}^2$. So:

$D_v f(0,0) = \nabla f(0,0)\cdot v$ and $D^2_{v,v} f(0,0) = v\cdot\mathbb{H}(0,0)\cdot v^T$.

We get $\nabla f(x,y) = (e^y - y\,e^x\,; x\,e^y - e^x) \Rightarrow \nabla f(0,0) = (1,\,-1)$;

so $\mathcal{D}_v f(0,0) = (1,\,-1)\cdot(\cos\alpha, \sin\alpha) = \cos\alpha - \sin\alpha$.

From $\nabla f(x,y) = (e^y - y\,e^x\,; x\,e^y - e^x)$ we get $\mathbb{H}(x,y) = \left\|\begin{matrix} -\,y\,e^x & e^y - e^x \\ e^y - e^x & x\,e^y \end{matrix}\right\|$ and so:

$$\mathbb{H}(0,0) = \left\|\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right\| \Rightarrow D^2_{v,v} f(0,0) = \|\cos\alpha \quad \sin\alpha\| \cdot \left\|\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}\right\| \cdot \left\|\begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix}\right\| = 0.$$

Finally $\mathcal{D}_v f(0,0) = \mathcal{D}^2_{vv} f(0,0) \Rightarrow \cos\alpha - \sin\alpha = 0 \Rightarrow \cos\alpha = \sin\alpha \Rightarrow \alpha = \dfrac{\pi}{4}$ or $\alpha = \dfrac{5\pi}{4}$.

II M 2) Given the system $\begin{cases} f(x,y,z) = xyz - x + y - z = 0 \\ g(x,y,z) = e^{x-y} - e^{y-z} = 0 \end{cases}$ and the point $\mathrm{P}_0 = (1,1,1)$, determine at least one implicit function that can be defined with it and then calculate the derivatives of such function at the proper point.

$\begin{cases} f(1,1,1) = 1-1+1-1 = 0 \\ g(1,1,1) = 1-1 = 0 \end{cases}$ and the system is satisfied.

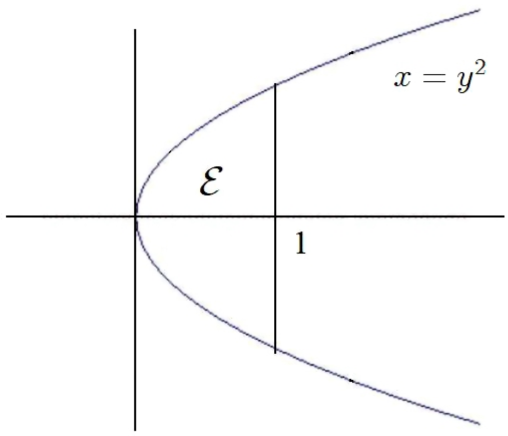
From $\dfrac{\partial(f,g)}{\partial(x,y,z)} = \left\|\begin{matrix} yz-1 & xz+1 & xy-1 \\ e^{x-y} & -\,e^{x-y} - e^{y-z} & e^{y-z} \end{matrix}\right\|$ we get:

$\dfrac{\partial(f,g)}{\partial(x,y,z)}(1,1,1) = \left\|\begin{matrix} 0 & 2 & 0 \\ 1 & -2 & 1 \end{matrix}\right\|$ and since $\left|\begin{matrix} 2 & 0 \\ -2 & 1 \end{matrix}\right| = 2 \neq 0$ it is possible to define an implicit function $x \to (y(x), z(x))$. For its derivatives we get:

$$\frac{dy}{dx} = -\,\frac{\left|\begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix}\right|}{\left|\begin{matrix} 2 & 0 \\ -2 & 1 \end{matrix}\right|} = -\,\frac{0}{2} = 0;\quad \frac{dz}{dx} = -\,\frac{\left|\begin{matrix} 2 & 0 \\ -2 & 1 \end{matrix}\right|}{\left|\begin{matrix} 2 & 0 \\ -2 & 1 \end{matrix}\right|} = -\,\frac{2}{2} = -\,1.$$

II M 3) Solve the problem: $\begin{cases} \text{Max/min } f(x,y) = 2x - y^2 \\ \text{u.c.}: y^2 \le x \le 1 \end{cases}$.

The objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a compact set, and so surely exist maximum and minimum values.



Using Kuhn-Tucker's conditions, we form the Lagrangian function:
$\Lambda(x,y,\lambda_1,\lambda_2) = 2x - y^2 - \lambda_1\left(y^2 - x\right) - \lambda_2\left(x-1\right).$

1) case $\lambda_1 = 0, \lambda_2 = 0$:

$\begin{cases} \Lambda'_x = 2 \neq 0 \\ \Lambda'_y = -2y = 0 \\ y^2 \leq x \\ x \leq 1 \end{cases}$: no solution.

2) case $\lambda_1 \neq 0, \lambda_2 = 0$:

$\begin{cases} \Lambda'_x = 2 + \lambda_1 = 0 \\ \Lambda'_y = -2y - 2\lambda_1 y = -2y(1+\lambda_1) = 0 \\ y^2 = x \\ x \leq 1 \end{cases}$ from which we get two systems:

$\begin{cases} x = 0 \\ y = 0 \\ \lambda_1 = -2 \\ x \leq 1 : \text{true} \end{cases}$ and since $\lambda_1 < 0$ the point $(0,0)$ may be a minimum point;

while the second system $\begin{cases} \lambda_1 = -2 \\ \lambda_1 = -1 \\ y^2 = x \\ x \leq 1 \end{cases}$ is impossible.

3) case $\lambda_1 = 0, \lambda_2 \neq 0$:

$\begin{cases} \Lambda'_x = 2 - \lambda_2 = 0 \\ \Lambda'_y = -2y = 0 \\ x = 1 \\ y^2 \leq x \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda_2 = 2 \\ 0 \leq 1 : \text{true} \end{cases}$. Since $\lambda_2 > 0$ the point $(1,0)$ may be a maximum point.

4) case $\lambda_1 \neq 0, \lambda_2 \neq 0$:

$\begin{cases} \Lambda'_x = 2 + \lambda_1 - \lambda_2 = 0 \\ \Lambda'_y = -2y - 2\lambda_1 y = 0 \\ x = y^2 \\ x = 1 \end{cases} \Rightarrow \begin{cases} \lambda_1 - \lambda_2 = -2 \\ -2y(1+\lambda_1) = 0 \\ x = 1 \\ y = 1 \end{cases}$ and $\begin{cases} \lambda_1 - \lambda_2 = -2 \\ -2y(1+\lambda_1) = 0 \\ x = 1 \\ y = -1 \end{cases}$.

$$\begin{cases} \lambda_1 - \lambda_2 = -2 \\ 1 + \lambda_1 = 0 \\ x = 1 \\ y = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \\ \lambda_1 = -1 \\ \lambda_2 = 1 \end{cases}$$

. Since $\lambda_1 < 0$ and $\lambda_2 > 0$ the point $(1,1)$ is nor a maximum nor a minimum point.

$$\begin{cases} \lambda_1 - \lambda_2 = -2 \\ 1 + \lambda_1 = 0 \\ x = 1 \\ y = -1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -1 \\ \lambda_1 = -1 \\ \lambda_2 = 1 \end{cases}$$

. Since $\lambda_1 < 0$ and $\lambda_2 > 0$ the point $(1,-1)$ is nor a maximum nor a minimum point.

So, from Weierstrass Theorem, $(1,0)$ is the maximum point with $f(1,0) = 2$ while $(0,0)$ is the minimum point with $f(0,0) = 0$.

II M 4) Given the function $f(x,y) = x^3 - 3xy + y^2$ analyze the nature of its stationary points.
To analyze the nature of the stationary points of the function we apply first and second order conditions. For the first order conditions we pose:

$$\nabla f(x,y) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 3x^2 - 3y = 3(x^2 - y) = 0 \\ f'_y = 2y - 3x = 0 \end{cases} \Rightarrow \begin{cases} x^2 - \frac{3}{2}x = x\left(x - \frac{3}{2}\right) = 0 \\ y = \frac{3}{2}x \end{cases}$$

and so we get two stationary points: $P_1 = (0;0)$ and $P_2 = \left(\dfrac{3}{2};\dfrac{9}{4}\right)$. For the second order conditions we construct the Hessian matrix: $\mathbb{H}(x,y) = \begin{Vmatrix} 6x & -3 \\ -3 & 2 \end{Vmatrix}$.

$\mathbb{H}(0;0) = \begin{Vmatrix} 0 & -3 \\ -3 & 2 \end{Vmatrix}$. Since $|\mathbb{H}_2| = -9 < 0$ the point $(0;0)$ is a saddle point.

$\mathbb{H}\left(\dfrac{3}{2};\dfrac{9}{4}\right) = \begin{Vmatrix} 9 & -3 \\ -3 & 2 \end{Vmatrix}$. Since $\begin{cases} |\mathbb{H}_1| = 9 > 0\,;\, 2 > 0 \\ |\mathbb{H}_2| = 18 - 9 > 0 \end{cases}$ the point P_2 is a minimum point.