TASK MATHEMATICS for ECONOMIC APPLICATIONS 19/09/2019

I M 1) Calcolate $\sqrt{\dfrac{2(1-i)(2+3i)}{(5+i)(1+i)}}$. Since

$$\frac{2(1-i)(2+3i)}{(5+i)(1+i)} = \frac{2(2-2i+3i+3)}{5+5i+i-1} = \frac{2(5+i)}{4+6i} = \frac{5+i}{2+3i} = \frac{5+i}{2+3i}\cdot\frac{2-3i}{2-3i} =$$

$$= \frac{10+2i-15i+3}{4+9} = \frac{13-13i}{13} = 1-i = \sqrt{2}\cdot\left(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) =$$

$= \sqrt{2}\cdot\left(\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}\right)$, we get:

$$\sqrt{\frac{2(1-i)(2+3i)}{(5+i)(1+i)}} = \sqrt{1-i} = \sqrt[4]{2}\left(\cos\left(\frac{7\pi}{8}+k\,\frac{2\pi}{2}\right) + i\sin\left(\frac{7\pi}{8}+k\,\frac{2\pi}{2}\right)\right), 0 \le k \le 1\,.$$

And so $c_1 = \sqrt[4]{2}\cdot\left(\cos\dfrac{7\pi}{8} + i\sin\dfrac{7\pi}{8}\right)$ and $c_2 = \sqrt[4]{2}\cdot\left(\cos\dfrac{15\pi}{8} + i\sin\dfrac{15\pi}{8}\right)$.

I M 2) Find eigenvalues and corresponding eigenvectors of the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 2 & 1 \\ 4 & 0 & 0 \\ 1 & 2 & 1 \end{Vmatrix}$.

From $|\mathbb{A} - \lambda\,\mathbb{I}| = 0$ we get $\begin{vmatrix} 1-\lambda & 2 & 1 \\ 4 & -\lambda & 0 \\ 1 & 2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 2 & 1 \\ 4 & -\lambda & 0 \\ \lambda & 2 & 1-\lambda \end{vmatrix} =$

$= 1\left(8+\lambda^2\right) + (1-\lambda)\left(\lambda^2-8\right) = -\lambda^3 + 2\lambda^2 + 8\lambda = 0 \Rightarrow -\lambda\left(\lambda^2 - 2\lambda - 8\right) = 0\,.$

So $\lambda_1 = 0$ and $\lambda_{2,3} = 1 \pm \sqrt{1+8} = 1 \pm 3 \Rightarrow \lambda_2 = 4,\ \lambda_3 = -2\,.$

For finding the corresponding eigenvectors we solve three systems.

$$\lambda_1 = 0 : \|\mathbb{A} - 0\cdot\mathbb{I}\|\cdot\mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 1 & 2 & 1 \\ 4 & 0 & 0 \\ 1 & 2 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 4x_1 = 0 \end{cases} \Rightarrow$$

$\Rightarrow \begin{cases} x_1 = 0 \\ x_3 = -2x_2 \end{cases}$ and so eigenvectors corresponing to $\lambda_1 = 0$ are $\mathbb{X}_1 = (0, x, -2x)$.

$$\lambda_2 = 4 : \|\mathbb{A} - 4\cdot\mathbb{I}\|\cdot\mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} -3 & 2 & 1 \\ 4 & -4 & 0 \\ 1 & 2 & -3 \end{Vmatrix}\cdot\begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow$$

$\Rightarrow \begin{cases} -3x_1 + 2x_2 + x_3 = 0 \\ 4x_1 - 4x_2 = 0 \\ x_1 + 2x_2 - 3x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = x_1 \\ x_2 = x_1 \\ x_3 = x_1 \end{cases}$ and so eigenvectors corresponing to $\lambda_2 = 4$ are $\mathbb{X}_2 = (x, x, x)$.

$$\lambda_3 = -2 : \|\mathbb{A} + 2\cdot\mathbb{I}\|\cdot\mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 3 & 2 & 1 \\ 4 & 2 & 0 \\ 1 & 2 & 3 \end{Vmatrix}\cdot\begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow$$

$\Rightarrow \begin{cases} 3x_1 + 2x_2 + x_3 = 0 \\ 4x_1 + 2x_2 = 0 \\ x_1 + 2x_2 + 3x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = x_1 \\ x_2 = -2x_1 \\ x_3 = x_1 \end{cases}$ and so eigenvectors corresponing to $\lambda_3 = -2$ are $\mathbb{X}_3 = (x, -2x, x)$.

I M 3) Consider the linear map $f:\mathbb{R}^4\to\mathbb{R}^3$, $\mathbb{Y}=\mathbb{A}\cdot\mathbb{X}$ for which:
$f(x_1,x_2,x_3,x_4)=(x_1+x_2+x_3;x_1+x_3+x_4;x_1+x_2+x_4)$.
Determine a basis for the Kernel and a basis for the Image of this linear map.

We have $\mathbb{A}=\begin{Vmatrix}1&1&1&0\\1&0&1&1\\1&1&0&1\end{Vmatrix}$. By elementary operations on the rows:

$(R_2\leftarrow R_2-R_1)$ and $(R_3\leftarrow R_3-R_1)$ we get : $\begin{Vmatrix}1&1&1&0\\0&-1&0&1\\0&0&-1&1\end{Vmatrix}$ from which we see that $\text{Rank}(\mathbb{A})=3$ and so $\text{Dim}(\text{Imm}(\mathbb{A}))=3$ and $\text{Dim}(\text{Ker}(\mathbb{A}))=4-3=1$.

To find a basis for the Kernel we must solve the system:

$$\mathbb{A}\cdot\mathbb{X}=\mathbb{O}\Rightarrow\begin{Vmatrix}1&1&1&0\\1&0&1&1\\1&1&0&1\end{Vmatrix}\cdot\begin{Vmatrix}x_1\\x_2\\x_3\\x_4\end{Vmatrix}=\begin{Vmatrix}0\\0\\0\end{Vmatrix}\Rightarrow\begin{cases}x_1+x_2+x_3=0\\x_1+x_3+x_4=0\\x_1+x_2+x_4=0\end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases}x_1+x_2+x_3=0\\-x_2+x_4=0\\-x_3+x_4=0\end{cases}\Rightarrow\begin{cases}x_1=-2x_4\\x_2=x_4\\x_3=x_4\end{cases}$$. So the vectors belonging to the Kernel are:

$\mathbb{X}=(-2x,x,x,x)$ and a basis may be the vector $\mathbb{X}=(-2,1,1,1)$.

To find a basis for the Image theoretically we should apply Rouchè-Capelli theorem to the system:

$$\mathbb{A}\cdot\mathbb{X}=\mathbb{Y}\Rightarrow\begin{Vmatrix}1&1&1&0\\1&0&1&1\\1&1&0&1\end{Vmatrix}\cdot\begin{Vmatrix}x_1\\x_2\\x_3\\x_4\end{Vmatrix}=\begin{Vmatrix}y_1\\y_2\\y_3\end{Vmatrix}\Rightarrow\begin{cases}x_1+x_2+x_3=y_1\\x_1+x_3+x_4=y_2\\x_1+x_2+x_4=y_3\end{cases}$$ and then we should study the Rank of the matrix and the Rank of the augmented matrix:

$\begin{Vmatrix}1&1&1&0&|&y_1\\1&0&1&1&|&y_2\\1&1&0&1&|&y_3\end{Vmatrix}$. But since $\text{Rank}(\mathbb{A})=3=\text{Dim}(\text{Imm}(\mathbb{A}))$, the map is a surjective one and so a basis for the Image is simply the standard basis of $\mathbb{R}^3$

I M 4) Since the vectors $\mathbb{X}_1=(1,2,-1)$, $\mathbb{X}_2=(2,0,1)$ e $\mathbb{X}_3=(x_1,x_2,x_3)$ form a basis for $\mathbb{R}^3$ and since in this basis the coordinates of the vector $\mathbb{Y}=(1,3,0)$ are $(2,-2,1)$, determine $\mathbb{X}_3$.

To satisfy what is required, we must solve the system:

$$\begin{Vmatrix}1&2&x_1\\2&0&x_2\\-1&1&x_3\end{Vmatrix}\cdot\begin{Vmatrix}2\\-2\\1\end{Vmatrix}=\begin{Vmatrix}1\\3\\0\end{Vmatrix}\Rightarrow\begin{cases}2-4+x_1=1\\4+0+x_2=3\\-2-2+x_3=0\end{cases}\Rightarrow\begin{cases}x_1=3\\x_2=-1\\x_3=4\end{cases}.$$

And so $\mathbb{X}_3=(3,-1,4)$.

II M 1) Given the equation $f(x,y)=x^3y+xy^2-2y=0$ and the point $\text{P}_0=(1,1)$ satisfying it, determine first and second order derivatives of the implicit function $x\to y(x)$ definable with it.

From $\nabla f(x,y)=\left(3x^2y+y^2;x^3+2xy-2\right)$ we get $\nabla f(1,1)=(4;1)$ and so:

$y'(1)=-\dfrac{4}{1}=-4$. Then we have:

$\mathbb{H}(x,y)=\begin{Vmatrix}6xy&3x^2+2y\\3x^2+2y&2x\end{Vmatrix}\Rightarrow\mathbb{H}(1;1)=\begin{Vmatrix}6&5\\5&2\end{Vmatrix}$ and so, since:

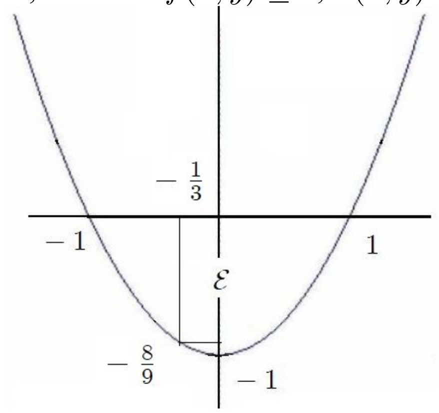
$$y'' = -\frac{f''_{xx} + 2f''_{xy}\,y' + f''_{yy}\,(y')^2}{f'_y}$$, we get $y''(1) = -\dfrac{6 + 10\cdot(-4) + 2(-4)^2}{1} = 2$.

II M 2) Given the function $f(x,y) = x\,y^3 - 3x^2 + 3x - 2y$ and the point $P_0 = (1,\,-2)$, compute $\mathcal{D}_v f(0,0)$, where v represents the direction from the origin $(0,0)$ to P_0.
$f(x,y) = x\,y^3 - 3x^2$ is a differentiable function $\forall\,(x,y) \in \mathbb{R}^2$.
So $D_v f(0,0) = \nabla f(0,0)\cdot v$.
We get $\nabla f(x,y) = \left(y^3 - 6x + 3\,; 3xy^2 - 2\right) \Rightarrow \nabla f(0,0) = (3,\,-2)$.
Since $\|P_0\| = \sqrt{1+4} = \sqrt{5}$ it is $v = \left(\dfrac{1}{\sqrt{5}},\,-\dfrac{2}{\sqrt{5}}\right)$ and therefore we have:

$$\mathcal{D}_v f(0,0) = \nabla f(0,0)\cdot v = (3,\,-2)\cdot\left(\frac{1}{\sqrt{5}},\,-\frac{2}{\sqrt{5}}\right) = \frac{7}{\sqrt{5}}.$$

II M 3) Solve the problem: $\begin{cases} \text{Max/min } f(x,y) = y\,(x-1) \\ \text{u.c.: } \begin{cases} x^2 - y - 1 \le 0 \\ y \le 0 \end{cases} \end{cases}$.

The objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a compact set, and so surely exist maximum and minimum values.
Since in $\mathcal{E}$ it is $y \le 0$ and $x \le 1$, it is also $f(x,y) \ge 0, \forall\,(x,y) \in \mathcal{E}$.



Using Kuhn-Tucker's conditions, we form the Lagrangian function:
$\Lambda(x,y,\lambda_1,\lambda_2) = yx - y - \lambda_1\left(x^2 - y - 1\right) - \lambda_2\,(y)$.
1) case $\lambda_1 = 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = y = 0 \\ \Lambda'_y = x - 1 = 0 \\ x^2 - y - 1 \le 0 \\ y \le 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ x^2 - y - 1 \le 0 \\ y \le 0 \end{cases}$$; but $\mathbb{H}(x;y) = \mathbb{H}(1;0) = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$ which indicates a saddle point, to be re-checked anyway since $(1;0)$ is on the boundary of $\mathcal{E}$.

2) case $\lambda_1 \ne 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = y - 2\lambda_1 x = 0 \\ \Lambda'_y = x - 1 + \lambda_1 = 0 \\ y = x^2 - 1 \\ y \le 0 \end{cases} \Rightarrow \begin{cases} x = 1 - \lambda_1 \\ y = (1-\lambda_1)^2 - 1 \\ (1-\lambda_1)^2 - 1 - 2\lambda_1(1-\lambda_1) = 0 \\ y \le 0 \end{cases}$$ which leads to the equation:

$\lambda_1^2 - 2\lambda_1 + 1 - 1 - 2\lambda_1 + 2\lambda_1^2 = 3\lambda_1^2 - 4\lambda_1 = \lambda_1(3\lambda_1 - 4) = 0$
from which we get two solutions:

$$\begin{cases} x = 1 \\ y = 0 \\ \lambda_1 = 0 \\ y \leq 0; \text{ true} \end{cases} \text{ already studied and } \begin{cases} x = -\frac{1}{3} \\ y = -\frac{8}{9} \\ \lambda_1 = \frac{4}{3} > 0 \\ y \leq 0; \text{ true} \end{cases} \text{, possible point of Max.}$$

3) case $\lambda_1 = 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = y = 0 \\ \Lambda'_y = x - 1 - \lambda_2 = 0 \\ y = 0 \\ x^2 - y - 1 \leq 0 \end{cases}$$

. But if $y = 0$ it is $f(x;0) = 0$ i.e. a constant function.

4) case $\lambda_1 \neq 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = y - 2\lambda_1 x = 0 \\ \Lambda'_y = x - 1 + \lambda_1 - \lambda_2 = 0 \\ y = x^2 - 1 \\ y = 0 \end{cases} \Rightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = -2 < 0 \\ x = -1 \\ y = 0 \end{cases} \text{ possible point of Min and}$$

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ x = 1 \\ y = 0 \end{cases}$$

already studied. So by Weierstrass's Theorem $\left(-\frac{1}{3}; -\frac{8}{9}\right)$ is the Maximum point, with $f\left(-\frac{1}{3}; -\frac{8}{9}\right) = \frac{32}{27}$ while $(-1;0)$, and all the points of the axis $y = 0$, are Minimum points, with $f(x;0) = 0$.

II M 4) Given the function $f(x,y) = x^3 - kxy + y^2$ analyze the nature of its stationary points on varying the parameter k.

To analyze the nature of the stationary points of the function we apply first and second order conditions. For the first order conditions we pose:

$$\nabla f(x,y) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 3x^2 - ky = 0 \\ f'_y = 2y - kx = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - \frac{k^2}{2}x = x\left(3x - \frac{k^2}{2}\right) = 0 \\ y = \frac{k}{2}x \end{cases}$$

and so we get two stationary points: $P_1 = (0;0)$ and $P_2 = \left(\frac{k^2}{6}; \frac{k^3}{12}\right)$.

For the second order conditions we construct the Hessian matrix: $\mathbb{H}(x,y) = \begin{Vmatrix} 6x & -k \\ -k & 2 \end{Vmatrix}$.

Since $\mathbb{H}(0;0) = \begin{Vmatrix} 0 & -k \\ -k & 2 \end{Vmatrix}$, and since $|\mathbb{H}_2| = -k^2 < 0$ the point $P_1 = (0;0)$ it is surely a saddle point if $k \neq 0$; if $k = 0$ we get $f(x,y) = x^3 + y^2$ and since $f(x,0) < 0$ for $x < 0$ while $f(x,0) > 0$ for $x > 0$, also if $k = 0$ the point $P_1 = (0;0)$ is a saddle point.

Then we study:

$\mathbb{H}\left(\frac{k^2}{6}; \frac{k^3}{12}\right) = \begin{Vmatrix} k^2 & -k \\ -k & 2 \end{Vmatrix}$, and since $\begin{cases} |\mathbb{H}_1| = k^2 \geq 0 \,;\, 2 > 0 \\ |\mathbb{H}_2| = 2k^2 - k^2 \geq 0 \end{cases}$ the point P_2 is surely a minimum point if $k \neq 0$. If $k = 0$ we get the case, already studied, of the point $(0,0)$.