TASK MATHEMATICS for ECONOMIC APPLICATIONS 12/10/2019

I M 1) Calcolate $\sqrt{(1+i)^4(1-i)^6}$. Since
$1+i=\sqrt{2}\cdot\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$, we get $(1+i)^4=4\left(\cos\pi+i\sin\pi\right)=-4$;
$1-i=\sqrt{2}\cdot\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$, we get $(1-i)^6=8\left(\cos\frac{21\pi}{2}+i\sin\frac{21\pi}{2}\right)=$
$=8\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=8i$. So $\sqrt{(1+i)^4(1-i)^6}=\sqrt{-32i}$.
Since $-32i=32\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$ we get:
$\sqrt{-32i}=\sqrt{32}\cdot\left(\cos\left(\frac{3\pi}{4}+k\,\frac{2\pi}{2}\right)+i\sin\left(\frac{3\pi}{4}+k\,\frac{2\pi}{2}\right)\right), 0\le k\le 1$ from which we get:
$c_1=4\sqrt{2}\cdot\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=4\sqrt{2}\left(-\frac{1}{\sqrt{2}}+i\,\frac{1}{\sqrt{2}}\right)=-4+4i$ and
$c_2=4\sqrt{2}\cdot\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)=4\sqrt{2}\left(\frac{1}{\sqrt{2}}-i\,\frac{1}{\sqrt{2}}\right)=4-4i$.

I M 2) Find the eigenvalues of the matrix $\mathbb{A}=\left\|\begin{matrix}0&0&1&1\\0&0&1&1\\1&1&0&0\\1&1&0&0\end{matrix}\right\|$.

From $|\mathbb{A}-\lambda\,\mathbb{I}|=0$ we get $\begin{vmatrix}-\lambda&0&1&1\\0&-\lambda&1&1\\1&1&-\lambda&0\\1&1&0&-\lambda\end{vmatrix}=\begin{vmatrix}-\lambda&0&1&1\\0&-\lambda&1&1\\1&1&-\lambda&0\\0&0&\lambda&-\lambda\end{vmatrix}=$

$=(-\lambda)\begin{vmatrix}-\lambda&1&1\\1&-\lambda&0\\0&\lambda&-\lambda\end{vmatrix}+(1)\begin{vmatrix}0&1&1\\-\lambda&1&1\\0&\lambda&-\lambda\end{vmatrix}=$

$=(-\lambda)\begin{vmatrix}-\lambda&2&1\\1&-\lambda&0\\0&0&-\lambda\end{vmatrix}+(1)(\lambda)(-\lambda-\lambda)=(-\lambda)(-\lambda)\big(\lambda^2-2\big)-2\lambda^2=$

$=(-\lambda)(-\lambda)\big(\lambda^2-2\big)-2\lambda^2=\lambda^4-4\lambda^2=\lambda^2\big(\lambda^2-4\big)=0$ and so eigenvalues are:
$\lambda_1=\lambda_2=0, \lambda_3=2, \lambda_4=-2$.

I M 3) Given the linear map $f:\mathbb{R}^4\to\mathbb{R}^3$, $\mathbb{Y}=\mathbb{A}\cdot\mathbb{X}$, with $\mathbb{A}=\left\|\begin{matrix}1&1&2&1\\1&-1&m&0\\1&-1&1&k\end{matrix}\right\|$, determine a basis for the Kernel and a basis for the Image of the linear map generated by $\mathbb{A}$, knowing that the Kernel and the Image have the same dimensions..
From Sylvester Theorem, it is $\text{Dim(Imm)}=\text{Rank}(\mathbb{A})$ and $\text{Dim (Ker)}=n-\text{Rank}(\mathbb{A})$.
Now $n=4$ and so $\text{Dim(Imm)}=\text{Dim (Ker)}=4-\text{Rank}(\mathbb{A})$ iff $\text{Rank}(\mathbb{A})=2$.
By elementary operations on the rows and $(R_2\leftarrow R_2-R_1)$ and $(R_3\leftarrow R_3-R_2)$ we get:

$$\begin{Vmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & m & 0 \\ 1 & -1 & 1 & k \end{Vmatrix} \to \begin{Vmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & m-2 & -1 \\ 0 & 0 & 1-m & k \end{Vmatrix}$$ from which we see that $\mathrm{Rank}(\mathbb{A}) = 2$ iff $m = 1$ and $k = 0$. So $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{Vmatrix}$.

To find a basis for the Kernel we must solve the system:

$$\mathbb{A} \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 + x_4 = 0 \\ x_1 - x_2 + x_3 = 0 \\ x_1 - x_2 + x_3 = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_1 + x_2 - 2x_1 + 2x_2 + x_4 = 0 \\ x_3 = -x_1 + x_2 \end{cases} \Rightarrow \begin{cases} x_3 = -x_1 + x_2 \\ x_4 = x_1 - 3x_2 \end{cases}.$$

So the vectors belonging to the Kernel are $\mathbb{X} = (x_1, x_2, -x_1 + x_2, x_1 - 3x_2)$ and a basis may be $\mathbb{X} = \{(1, 0, -1, 1); (0, 1, 1, -3)\}$.

To find a basis for the Image we must apply Rouchè-Capelli theorem to the system:

$$\mathbb{A} \cdot \mathbb{X} = \mathbb{Y} \Rightarrow \begin{Vmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix} = \begin{Vmatrix} y_1 \\ y_2 \\ y_3 \end{Vmatrix} \Rightarrow \begin{cases} x_1 + x_2 + 2x_3 + x_4 = y_1 \\ x_1 - x_2 + x_3 = y_2 \\ x_1 - x_2 + x_3 = y_3 \end{cases}$$ and then we should study the Rank of the matrix and the Rank of the augmented matrix:

$$\begin{Vmatrix} 1 & 1 & 2 & 1 & | & y_1 \\ 1 & -1 & 1 & 0 & | & y_2 \\ 1 & -1 & 1 & 0 & | & y_3 \end{Vmatrix}.$$

By elementary operations on the rows and $(R_2 \leftarrow R_2 - R_1)$ and $(R_3 \leftarrow R_3 - R_2)$ we get:

$$\begin{Vmatrix} 1 & 1 & 2 & 1 & | & y_1 \\ 1 & -1 & 1 & 0 & | & y_2 \\ 1 & -1 & 1 & 0 & | & y_3 \end{Vmatrix} \to \begin{Vmatrix} 1 & 1 & 2 & 1 & | & y_1 \\ 0 & -2 & -1 & -1 & | & y_2 - y_1 \\ 0 & 0 & 0 & 0 & | & y_3 - y_2 \end{Vmatrix}$$ from which we get: $y_3 = y_2$ and so the vectors belonging to the Image are $\mathbb{Y} = (y_1, y_2, y_2)$ and a basis may be $\mathbb{Y} = \{(1, 0, 0); (0, 1, 1)\}$.

I M 4) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$ determine its inverse matrix.

Since $|\mathbb{A}| = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$ the matrix has its unique inverse matrix, which is given by the transpose of the adjugate matrix adj($\mathbb{A}$), divided by the determinant of $\mathbb{A}$.

It is $\mathrm{Adj}(\mathbb{A}) = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{Vmatrix}$, then $(\mathrm{Adj}(\mathbb{A}))^{\mathrm{T}} = \begin{Vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$, and since $\det(\mathbb{A}) = 1$, we finally obtain: $\mathbb{A}^{-1} = \begin{Vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}$.

II M 1) Given the equation $f(x, y) = x^3 y - x y^3 + x - y = 1$ and the point $\mathrm{P}_0 = (1, 0)$ satisfying it, determine first order derivative of a possible implicit function definable with it.

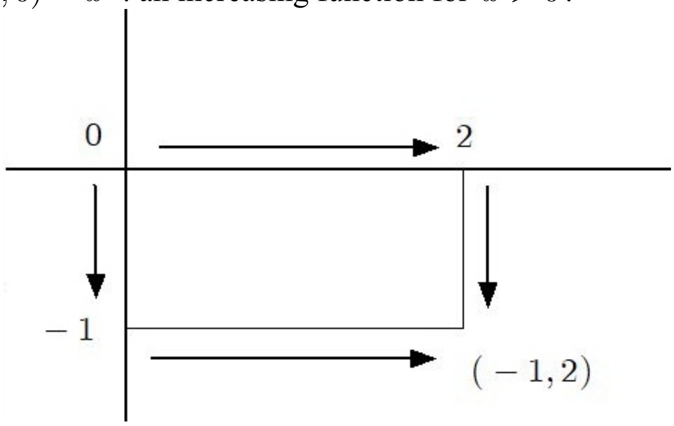
From $\nabla f(x,y)=\left(3x^2y-y^3+1;x^3-3xy^2-1\right)$ we get $\nabla f(1,0)=(1;0)$ and so it is possible to define only an implicit function $y\to x(y)$: $x'(0)=-\dfrac{0}{1}=0$.

II M 2) Given the functions $f(x,y)=x^2+y^2$ and $g(x,y)=x+xy$, the point $P_0=(1,1)$ and the unit vector $v=(\cos\alpha,\sin\alpha)$, find the values of α for which $\mathcal{D}_v\,f(P_0)=\mathcal{D}_v\,g(P_0)$.
$f(x,y)=x^2+y^2$ and $g(x,y)=x+xy$ are differentiable function $\forall\,(x,y)\in\mathbb{R}^2$.
So $D_vf(1,1)=\nabla f(1,1)\cdot v=D_vg(1,1)=\nabla g(1,1)\cdot v$. We get:
$\nabla f(x,y)=(2x\,;2y)\Rightarrow\nabla f(1,1)=(2,2)$, $\nabla g(x,y)=(1+y\,;x)\Rightarrow\nabla g(1,1)=(2,1)$.
So $D_vf(1,1)=\nabla f(1,1)\cdot(\cos\alpha,\sin\alpha)=D_vg(1,1)=\nabla g(1,1)\cdot(\cos\alpha,\sin\alpha)\Rightarrow$
$\Rightarrow(2,2)\cdot(\cos\alpha,\sin\alpha)=(2,1)\cdot(\cos\alpha,\sin\alpha)\Rightarrow 2\cos\alpha+2\sin\alpha=2\cos\alpha+\sin\alpha\Rightarrow$
$\sin\alpha=0\Rightarrow\alpha=0$ or $\alpha=\pi$.

II M 3) Determine maximum and minimum points for the function $f(x,y)=x^2-y$ in the rectangle $\mathbb{Q}=\left\{(x,y)\in\mathbb{R}^2\colon 0\le x\le 2\,;\,-1\le y\le 0\right\}$.
The objective function of the problem is a continuous function, the feasible region $\mathbb{Q}$ is a compact set, and so surely exist maximum and minimum values.
We don't use Kuhn-Tucker's conditions nor the Lagrangian function. Testing for free maximum and minimum points we get:

$$\nabla f(x,y)=\mathbb{O}\Rightarrow\begin{cases}f'_x=2x=0\\ f'_y=-1\neq 0\end{cases}$$ and so we get no solutions.

Now we study our function on the points of the boundary of $\mathbb{Q}$.
For $x=0$ we get $f(0,y)=-y$: a decreasing function;
for $x=2$ we get $f(2,y)=4-y$: a decreasing function;
for $y=-1$ we get $f(x,-1)=x^2+1$: an increasing function for $x>0$;
for $y=0$ we get $f(x,0)=x^2$: an increasing function for $x>0$.



So, using proper arrows, we see that $(0,0)$ is the minimum point, with $f(0,0)=0$ while $(2,-1)$ is the maximum point, with $f(2,-1)=5$.

II M 4) Given the function $f(x,y)=x^3-x^2y^2+y^2$ analyze the nature of its stationary points.
To analyze the nature of the stationary points of the function we apply first and second order conditions. For the first order conditions we pose:

$$\nabla f(x,y)=\mathbb{O}\Rightarrow\begin{cases}f'_x=3x^2-2xy^2=0\\ f'_y=-2x^2y+2y=0\end{cases}\Rightarrow\begin{cases}x(3x-2y^2)=0\\ 2y(1-x^2)=0\end{cases}$$ from which we get:

$$\begin{cases}x=0\\ y=0\end{cases}\text{ or }\begin{cases}x=0\\ 1-x^2=0\end{cases}\text{ or }\begin{cases}3x-2y^2=0\\ y=0\end{cases}\text{ or }\begin{cases}3x-2y^2=0\\ 1-x^2=0\end{cases}.$$

From the first system we get the point $(0,0)$; the second system is impossible.

From the third system we get another time the point $(0,0)$, and from the last system we get $\begin{cases} x = -1 \\ y^2 = \frac{3}{2}x \end{cases}$, an impossible system, and $\begin{cases} x = 1 \\ y^2 = \frac{3}{2}x \end{cases}$, from which we get the two solutions $\left(1, \sqrt{\frac{3}{2}}\right)$ and $\left(1, -\sqrt{\frac{3}{2}}\right)$.

For the second order conditions we construct the Hessian matrix:

$$\mathbb{H}(x,y) = \begin{Vmatrix} 6x - 2y^2 & -4xy \\ -4xy & 2 - 2x^2 \end{Vmatrix}.$$

Since $\mathbb{H}(0,0) = \begin{Vmatrix} 0 & 0 \\ 0 & 2 \end{Vmatrix}$ we cannot use the leading principle minors. But $f(0,0) = 0$ and, in the orizontal axis we have $f(x,0) = x^3$, which is negative for $x < 0$ and positive for $x > 0$; so $(0,0)$ is a saddle point.

Then $\mathbb{H}\left(1, \sqrt{\frac{3}{2}}\right) = \begin{Vmatrix} 3 & -4\sqrt{\frac{3}{2}} \\ -4\sqrt{\frac{3}{2}} & 0 \end{Vmatrix}$ and since $|\mathbb{H}_2| = -24 < 0$, $\left(1, \sqrt{\frac{3}{2}}\right)$ is a saddle point.

Finally $\mathbb{H}\left(1, -\sqrt{\frac{3}{2}}\right) = \begin{Vmatrix} 3 & 4\sqrt{\frac{3}{2}} \\ 4\sqrt{\frac{3}{2}} & 0 \end{Vmatrix}$ and since $|\mathbb{H}_2| = -24 < 0$, $\left(1, -\sqrt{\frac{3}{2}}\right)$ is another saddle point.