Prova Intermedia di Analisi Matematica del 16/12/2014 $\boxed{AM1}$

IM1) $z_1 = 1\cdot\left(\cos\frac{4}{3}\pi + i\,\text{sen}\,\frac{4}{3}\pi\right)$; $z_2 = 1\cdot\left(\cos\frac{5}{6}\pi + i\,\text{sen}\,\frac{5}{6}\pi\right)$; $z_3 = 1\cdot\left(\cos\frac{3}{2}\pi + i\,\text{sen}\,\frac{3}{2}\pi\right)$.

$z_1^2 = 1\cdot\left(\cos\frac{8}{3}\pi + i\,\text{sen}\,\frac{8}{3}\pi\right) = 1\cdot\left(\cos\frac{2}{3}\pi + i\,\text{sen}\,\frac{2}{3}\pi\right)$;

$z_2^3 = 1\cdot\left(\cos\frac{5}{2}\pi + i\,\text{sen}\,\frac{5}{2}\pi\right) = 1\cdot\left(\cos\frac{\pi}{2} + i\,\text{sen}\,\frac{\pi}{2}\right)$;

$z_3^3 = 1\cdot\left(\cos\frac{9}{2}\pi + i\,\text{sen}\,\frac{9}{2}\pi\right) = 1\cdot\left(\cos\frac{\pi}{2} + i\,\text{sen}\,\frac{\pi}{2}\right)$;

$z = \left(\cos\frac{\pi}{2} + i\,\text{sen}\,\frac{\pi}{2}\right)\cdot\left(\cos\left(\frac{2}{3}\pi + \frac{\pi}{2} - \frac{\pi}{2}\right) + i\,\text{sen}\left(\frac{2}{3}\pi + \frac{\pi}{2} - \frac{\pi}{2}\right)\right) = \cos\left(\frac{\pi}{2} + \frac{2}{3}\pi\right) + i\,\text{sen}\left(\frac{\pi}{2} + \frac{2}{3}\pi\right) =$

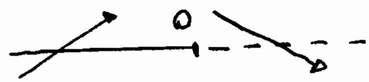
$= \cos\frac{7}{6}\pi + i\,\text{sen}\,\frac{7}{6}\pi$. $\sqrt{z} = 1\cdot\left(\cos\left(\frac{7}{12}\pi + k\frac{2\pi}{2}\right) + i\,\text{sen}\left(\frac{7}{12}\pi + k\cdot\frac{2\pi}{2}\right)\right)$; $0 \le k \le 1$.

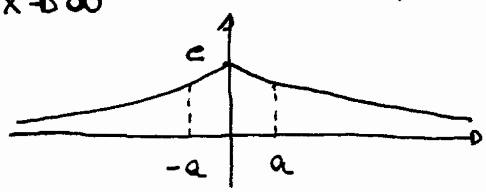
Per $k=0$: $\cos\frac{7}{12}\pi + i\,\text{sen}\,\frac{7}{12}\pi$; Per $k=1$: $\cos\frac{19}{12}\pi + i\,\text{sen}\,\frac{19}{12}\pi$.

IM2) $f_n(x) = e^{1-nx^2}$. C.E. $= \mathbb{R}$. $\lim_{n\to+\infty} e^{1-nx^2} = \begin{cases} 0: & x \neq 0 \\ e: & x = 0 \end{cases}$. $f_n(0) = e$.

Quindi $C = \mathbb{R}$ e $f(x) = \begin{cases} 0: & x \neq 0 \\ e: & x = 0 \end{cases}$. La convergenza non può essere uniforme in tutto $\mathbb{R}$.

$\lim_{x\to\infty} e^{1-nx^2} = 0^+$; $f_n'(x) = (-2nx)\cdot e^{1-nx^2} \geq 0$ per $x \leq 0$





Preso $a > 0$ risulta: $\sup_{x\in[a;+\infty[} \{|f_n(x) - f(x)|\} = f_n(a)$

e quindi $\lim_{n\to+\infty} \sup_{x\in[a;+\infty[} \{|f_n(x) - f(x)|\} = \lim_{n\to+\infty} f_n(a) = 0$.

Analogamente in ogni intervallo del tipo $]-\infty;-a]$. Quindi la successione di funzioni converge uniformemente in ogni intervallo del tipo $]-\infty;-a]$ e del tipo $[a;+\infty[$, con $a > 0$.

$\sum_{n=0}^{+\infty} e^{1-nx^2} = \sum_{n=0}^{+\infty} e\cdot\left(e^{-x^2}\right)^n = e\cdot\sum_{n=0}^{+\infty}\left(e^{-x^2}\right)^n$: Serie geometrica di ragione e^{-x^2}.

Risulta convergente se: $e^{-x^2} < 1 \Rightarrow x \neq 0$. $S(x) = e\cdot\frac{1}{1-e^{-x^2}} = \frac{e^{x^2+1}}{e^{x^2}-1}$.

AM2

IM3) $f(x;y)=\sqrt{x^2|x|}+\sqrt{y^2|y|}$. $f(0;0)=0=\lim_{(x;y)\to(0;0)} f(x;y)$; $f(x;y)\in\mathcal{C}(0;0)$.

$$\frac{\partial f}{\partial x}(0;0)=\lim_{h\to 0}\frac{\sqrt{(0+h)^2\cdot|0+h|}+\sqrt{0^2\cdot|0|}-0}{h}=\lim_{h\to 0}\frac{|h|\cdot\sqrt{|h|}}{h}=0;$$

$$\frac{\partial f}{\partial y}(0;0)=\lim_{h\to 0}\frac{\sqrt{0^2\cdot|0|}+\sqrt{(0+h)^2\cdot|0+h|}-0}{h}=\lim_{h\to 0}\frac{|h|\cdot\sqrt{|h|}}{h}=0.$$

$$\lim_{(x;y)\to(0;0)}\frac{f(x;y)-f(0;0)-\nabla f(0;0)\cdot(x-0;y-0)}{\sqrt{x^2+y^2}}=\lim_{(x;y)\to(0;0)}\frac{\sqrt{x^2|x|}+\sqrt{y^2|y|}}{\sqrt{x^2+y^2}}\Rightarrow$$

$$\Rightarrow\lim_{\rho\to 0}\frac{\sqrt{\rho^2\cos^2\vartheta\,|\rho\cos\vartheta|}+\sqrt{\rho^2\operatorname{sen}^2\vartheta\,|\rho\operatorname{sen}\vartheta|}}{\rho}=\lim_{\rho\to 0}\sqrt{\rho}\cdot\left(|\cos\vartheta|\sqrt{|\cos\vartheta|}+|\operatorname{sen}\vartheta|\sqrt{|\operatorname{sen}\vartheta|}\right)=$$

$=0$ in modo uniforme in quanto $\sqrt{\rho}\cdot\left(|\cos\vartheta|\cdot\sqrt{|\cos\vartheta|}+|\operatorname{sen}\vartheta|\sqrt{|\operatorname{sen}\vartheta|}\right)\leq\sqrt{\rho}(1+1)\leq 2\sqrt{\rho}$.

IM4) $f(x;y)=e^{x+y}-x+y=1$. $f(0;0)=1-0+0=1$.

$\nabla f(x;y)=\left(e^{x+y}-1;\ e^{x+y}+1\right)$; $\nabla f(0;0)=(1-1;1+1)=(0;2)$.

Si può definire una funzione implicita $y=y(x)$ in $\mathcal{I}(0)$.

Risulta $y'(0)=-\frac{0}{2}=0$, quindi si tratta di un punto stazionario.

$$H(x;y)=\begin{Vmatrix}e^{x+y} & e^{x+y}\\ e^{x+y} & e^{x+y}\end{Vmatrix};\ H(0;0)=\begin{Vmatrix}1 & 1\\ 1 & 1\end{Vmatrix}.$$

$$y''(0)=-\frac{f''_{xx}+2f''_{xy}\cdot y'+f''_{yy}\cdot(y')^2}{f'_y}=-\frac{1+2\cdot 1\cdot 0+1\cdot 0}{2}=-\frac{1}{2}<0.$$

Quindi $y''(0)<0$ e si tratta quindi di un punto di massimo.

AM3

IM5) $f(x;y)=x^2-xy+y^2$. Funzione continua e differenziabile due volte in $\mathbb{R}^2$.

$\mathcal{D}_v f(1;1) = \nabla f(1;1)\circ v$; $\mathcal{D}^2_{v,v} f(1;1) = v\cdot H(1;1)\cdot v^T$.

$\nabla f(x;y) = (2x-y\,;\,-x+2y)$; $\nabla f(1;1) = (2-1\,;\,-1+2) = (1\,;\,1)$.

$$H(x;y) = \begin{Vmatrix} 2 & -1 \\ -1 & 2 \end{Vmatrix} = H(1;1). \qquad v = (\cos\alpha\,;\,\operatorname{sen}\alpha)$$

$\mathcal{D}_v f(1;1) = (1;1)\cdot(\cos\alpha;\operatorname{sen}\alpha) = \cos\alpha + \operatorname{sen}\alpha$

$[\mathcal{D}_v f(1;1)]^2 = (\cos\alpha+\operatorname{sen}\alpha)^2 = 1+2\cos\alpha\operatorname{sen}\alpha = 1+\operatorname{sen}2\alpha$

$$\mathcal{D}^2_{v,v} f(1;1) = \begin{Vmatrix}\cos\alpha & \operatorname{sen}\alpha\end{Vmatrix}\cdot\begin{Vmatrix} 2 & -1 \\ -1 & 2 \end{Vmatrix}\cdot\begin{Vmatrix}\cos\alpha \\ \operatorname{sen}\alpha\end{Vmatrix} = \begin{Vmatrix}\cos\alpha & \operatorname{sen}\alpha\end{Vmatrix}\cdot\begin{Vmatrix} 2\cos\alpha-\operatorname{sen}\alpha \\ 2\operatorname{sen}\alpha-\cos\alpha \end{Vmatrix} =$$

$$= 2\cos^2\alpha - \operatorname{sen}\alpha\cos\alpha + 2\operatorname{sen}^2\alpha - \operatorname{sen}\alpha\cos\alpha = 2-2\operatorname{sen}\alpha\cos\alpha = 2-\operatorname{sen}2\alpha.$$

Quindi dovrà essere: $1+\operatorname{sen}2\alpha = 2-\operatorname{sen}2\alpha \Rightarrow$

$\Rightarrow 2\operatorname{sen}2\alpha = 1 \Rightarrow \operatorname{sen}2\alpha = \frac{1}{2} \Rightarrow 2\alpha = \operatorname{arcsen}\frac{1}{2} = \frac{\pi}{6} \Rightarrow$

$\Rightarrow \alpha = \frac{1}{2}\cdot\frac{\pi}{6} = \frac{\pi}{12}$.