Intermediate Test QUANTITATIVE METHODS
for ECONOMIC APPLICATIONS 21/11/2019

I M 1) Given the polynomial equation $x^2 - \sqrt{2}\,x + k = 0$, determine the value of k for which this equation admits the solution $x = e^{\frac{\pi}{4}i}$. Then calculate the square roots of the other solution.

From $x = e^{\frac{\pi}{4}i}$ we get $x = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}$; from $x^2 - \sqrt{2}\,x + k = 0$ we get:

$$x = \frac{\sqrt{2} \pm \sqrt{2-4k}}{2} = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2-4k}}{2} = \frac{\sqrt{2}}{2} \pm i\,\frac{\sqrt{2}}{2} \text{ if } 2-4k = -2 \Rightarrow k = 1\,.$$

So the second solution of the equation is $x_2 = \frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2} = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}$.

So $\sqrt{x_2} = \cos\left(\frac{7\pi}{8} + k\,\frac{2\pi}{2}\right) + i\sin\left(\frac{7\pi}{8} + k\,\frac{2\pi}{2}\right), 0 \le k \le 1$ from which:

$c_1 = \cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}$ and $c_2 = \cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & k & 0 \\ k & 1 & -1 \\ 0 & -1 & 1 \end{Vmatrix}$ determine the value of the parameter k such that the matrix admits the eigenvalue $\lambda = 0$ and, for this value of k, find one modal matrix which diagonalizes $\mathbb{A}$.

If the matrix admits the eigenvalue $\lambda = 0$ surely the matrix is a singolar one, i.e. its determinant is equal to zero. So $|\mathbb{A}| = \begin{vmatrix} 1 & k & 0 \\ k & 1 & -1 \\ 0 & -1 & 1 \end{vmatrix} = 1\,(1-1) - k\,(k-0) = -k^2 = 0 \text{ iff } k = 0$.

So $\mathbb{A} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{Vmatrix}$ and so, from $|\mathbb{A} - \lambda\,\mathbb{I}| = 0$ we get:

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)((1-\lambda)(1-\lambda) - 1) = (1-\lambda)\big(\lambda^2 - 2\lambda\big) = 0$$

to get the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$ and $\lambda_3 = 2$.
Since the matrix is a symmetric one, the matrix is diagonalizable. Furthermore it has all distinct eigenvalues. To get the corresponding eigenvectors, we must solve three systems.
For $\lambda_1 = 0$ we solve:

$$|\mathbb{A} - 0\cdot\mathbb{I}| = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x = 0 \\ y - z = 0 \\ -y + z = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = z \end{cases}$$

and so eigenvectors corresponding to $\lambda_1 = 0$ are $\mathbb{V}_1 = (0, y, y) \Rightarrow (0, 1, 1)$.
For $\lambda_2 = 1$ we solve:

$$|\mathbb{A} - 1\cdot\mathbb{I}| = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} \forall\, x \\ z = 0 \\ y = 0 \end{cases}$$

and so eigenvectors corresponding to $\lambda_2 = 1$ are $\mathbb{V}_2 = (x, 0, 0) \Rightarrow (1, 0, 0)$.

For $\lambda_3 = 2$ we solve:

$$|\mathbb{A} - 2\cdot\mathbb{I}| = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x = 0 \\ y + z = 0 \\ y + z = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ z = -y \end{cases}$$

and so eigenvectors corresponding to $\lambda_3 = 2$ are $\mathbb{V}_3 = (0, y, -y) \Rightarrow (0, 1, -1)$.
Since the matrix is a symmetric one, the matrix is diagonalizable by an orthogonal matrix, and so we must transform the three vectors in unit vectors, to get the modal orthogonal matrix which is:

$$\mathbb{U} = \begin{Vmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{Vmatrix}.$$

I M 3) If in the basis $\mathbb{V} = \{(1,1,0); (1,-1,0); (2,0,1)\}$ the vector $\mathbb{X}$ has coordinates $(1, 2, -1)$, determine its coordinates in the basis $\mathbb{W} = \{(1,1,0); (1,-1,0); (0,2,1)\}$.

If the vector $\mathbb{X}$ has coordinates $(1, 2, -1)$ in the basis $\mathbb{V} = \{(1,1,0); (1,-1,0); (2,0,1)\}$ we get: $\mathbb{X} = \begin{Vmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 2 \\ -1 \end{Vmatrix} = \begin{Vmatrix} 1 \\ -1 \\ -1 \end{Vmatrix}$. To determine the coordinates of $\mathbb{X}$ in the other basis $\mathbb{W} = \{(1,1,0); (1,-1,0); (0,2,1)\}$, we must solve the system:

$$\begin{Vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 1 \\ -1 \\ -1 \end{Vmatrix} \text{ or calculate } \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{Vmatrix}^{-1} \cdot \begin{Vmatrix} 1 \\ -1 \\ -1 \end{Vmatrix}.$$

Using the first method we get:

$$\begin{Vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 1 \\ -1 \\ -1 \end{Vmatrix} \Rightarrow \begin{cases} x + y = 1 \\ x - y + 2z = -1 \\ z = -1 \end{cases} \Rightarrow \begin{cases} x + y = 1 \\ x - y = 1 \\ z = -1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ z = -1 \end{cases}.$$

Using the second method we get: $\text{Adj}(\mathbb{A}) = \begin{Vmatrix} -1 & -1 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & -2 \end{Vmatrix}$.

Then $(\text{adj}(\mathbb{A}))^{\text{T}} = \begin{Vmatrix} -1 & -1 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & -2 \end{Vmatrix}$, and since $\det(\mathbb{A}) = -2$, we finally obtain:

$$\mathbb{A}^{-1} = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 0 & 1 \end{Vmatrix} \text{ and so we get } \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & -1 \\ \frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ -1 \\ -1 \end{Vmatrix} = \begin{Vmatrix} 1 \\ 0 \\ -1 \end{Vmatrix}.$$

I M 4) Given a linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathbb{Y} = \mathbb{A} \cdot \mathbb{X}$, determine the matrix $\mathbb{A}$ knowing that such linear map satisfies these three conditions:
a) $f(1,2,1) = (1,3,3)$;
b) $(1,1,0) \in \text{Ker}(f)$;
c) $(1,1,1)$ is an eigenvector of $\mathbb{A}$ corresponding to the eigenvalue $\lambda = 1$.

From the first condition we get:

$$\begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 2 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1 \\ 3 \\ 3 \end{Vmatrix} \Rightarrow \begin{cases} a_{11} + 2a_{12} + a_{13} = 1 \\ a_{21} + 2a_{22} + a_{23} = 3 \\ a_{31} + 2a_{32} + a_{33} = 3 \end{cases};$$

from the second condition we get:

$$\begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \\ 0 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} a_{11} + a_{12} = 0 \\ a_{21} + a_{22} = 0 \\ a_{31} + a_{32} = 0 \end{cases};$$

from the third condition we get:

$$\begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix} \Rightarrow \begin{cases} a_{11} + a_{12} + a_{13} = 1 \\ a_{21} + a_{22} + a_{23} = 1 \\ a_{31} + a_{32} + a_{33} = 1 \end{cases}.$$

We have so obtained three systems:

$$\begin{cases} a_{11} + 2a_{12} + a_{13} = 1 \\ a_{11} + a_{12} = 0 \\ a_{11} + a_{12} + a_{13} = 1 \end{cases} \Rightarrow \begin{cases} a_{11} = 0 \\ a_{12} = 0 \\ a_{13} = 1 \end{cases}$$

$$\begin{cases} a_{21} + 2a_{22} + a_{23} = 3 \\ a_{21} + a_{22} = 0 \\ a_{21} + a_{22} + a_{23} = 1 \end{cases} \Rightarrow \begin{cases} a_{21} = -2 \\ a_{22} = 2 \\ a_{23} = 1 \end{cases}$$

$$\begin{cases} a_{31} + 2a_{32} + a_{33} = 3 \\ a_{31} + a_{32} = 0 \\ a_{31} + a_{32} + a_{33} = 1 \end{cases} \Rightarrow \begin{cases} a_{31} = -2 \\ a_{32} = 2 \\ a_{33} = 1 \end{cases}$$

And finally $\mathbb{A} = \begin{Vmatrix} 0 & 0 & 1 \\ -2 & 2 & 1 \\ -2 & 2 & 1 \end{Vmatrix}$.

I M 5) Given the linear system $\begin{cases} x_1 - 2x_2 + 2x_3 + x_4 = 1 \\ 3x_1 - x_2 + 2x_4 = 1 \\ x_1 + 3x_2 + mx_3 + k\,x_4 = 0 \end{cases}$, check, depending on the parameters m and k, the existence and the number of its solutions.

To solve the problem we must apply Rouchè-Capelli theorem to the system, and so we study the Rank of the matrix $\mathbb{A}$ and the Rank of the Augmented matrix $(\mathbb{A}|\mathbb{Y})$.
By elementary operations on the rows $(R_2 \leftarrow R_2 - 3R_1)$ and $(R_3 \leftarrow R_3 - R_1)$ we get:

$$\begin{Vmatrix} 1 & -2 & 2 & 1 & | & 1 \\ 3 & -1 & 0 & 2 & | & 1 \\ 1 & 3 & m & k & | & 0 \end{Vmatrix} \to \begin{Vmatrix} 1 & -2 & 2 & 1 & | & 1 \\ 0 & 5 & -6 & -1 & | & -2 \\ 0 & 5 & m-2 & k-1 & | & -1 \end{Vmatrix};$$

by another elementary operation on the rows $(R_3 \leftarrow R_3 - R_2)$ we get:

$$\begin{Vmatrix} 1 & -2 & 2 & 1 & | & 1 \\ 0 & 5 & -6 & -1 & | & -2 \\ 0 & 5 & m-2 & k-1 & | & -1 \end{Vmatrix} \to \begin{Vmatrix} 1 & -2 & 2 & 1 & | & 1 \\ 0 & 5 & -6 & -1 & | & -2 \\ 0 & 0 & m+4 & k & | & 1 \end{Vmatrix}.$$

Using this last matrix we see that:
If $m = -4$ and $k = 0$ it is $\text{Rank}(\mathbb{A}) = 2 < \text{Rank}(\mathbb{A}|\mathbb{Y}) = 3$ and so the system has no solutions;
if $m \neq -4$ or $k \neq 0$ it is $\text{Rank}(\mathbb{A}) = 3 = \text{Rank}(\mathbb{A}|\mathbb{Y})$ and so the system has ∞^1 solutions.