Intermediate test Mathematics for Economic Applications 13/12/2014

I M 1) To answer to the two questions the first step is to calculate the characteristic polynomial of the matrix $\mathbb{A}$: $p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & k \\ 0 & -1 & 1-\lambda \end{vmatrix} =$

$= (1-\lambda)\begin{vmatrix} -\lambda & k \\ -1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1 & k \\ 0 & 1-\lambda \end{vmatrix} =$

$= (1-\lambda)(-\lambda(1-\lambda)+k) - (1-\lambda) = (1-\lambda)\left(\lambda^2 - \lambda + k - 1\right).$

The three eingenvalues of $\mathbb{A}$ are $\lambda_1 = 1$, $\lambda_{2,3} = \dfrac{1 \pm \sqrt{1-4(k-1)}}{2} = \dfrac{1 \pm \sqrt{5-4k}}{2}$.

The matrix may be not diagonalizable when it has at least a multiple eingenvalue (with algebraic multiplicity greater than 1), and this is possible in two cases:

when $5-4k=0$ or when $\dfrac{1 \pm \sqrt{5-4k}}{2} = 1$.

- $5-4k=0$ implies $k = \dfrac{5}{4}$;

- $\dfrac{1 \pm \sqrt{5-4k}}{2} = 1$ can be rewritten as $\pm\sqrt{5-4k} = 1 \Rightarrow 5-4k = 1 \Rightarrow k = 1$.

For $k = \dfrac{5}{4}$ we get $\lambda_2 = \lambda_3 = \dfrac{1}{2}$ and so $\left\|\mathbb{A} - \dfrac{1}{2}\cdot\mathbb{I}\right\| = \left\|\begin{matrix} \frac{1}{2} & 1 & 0 \\ 1 & -\frac{1}{2} & \frac{5}{4} \\ 0 & -1 & \frac{1}{2} \end{matrix}\right\|$;

it's very easy to see that $\left\|\mathbb{A} - \dfrac{1}{2}\cdot\mathbb{I}\right\|$ has rank equal to 2 and thus the dimension of the eigenspace associated to the eigenvalue $\lambda = \dfrac{1}{2}$ is 1, less than the algebraic multiplicity, and so $\mathbb{A}$ is not diagonalizable.

For $k=1$, we get $\lambda_1 = \lambda_2 = 1$ and $\|\mathbb{A} - 1\cdot\mathbb{I}\| = \left\|\begin{matrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & -1 & 0 \end{matrix}\right\|$; as before it's easy to see that the matrix $\|\mathbb{A} - 1\cdot\mathbb{I}\|$ has rank equal to 2 and thus the dimension of the eigenspace associated to the eingenvalue $\lambda = 1$ is 1, less than the algebraic multiplicity.

$\mathbb{A}$ is not diagonalizable when $k=1$ and when $k = \dfrac{5}{4}$.

For the second question note that:

$\lambda = \cos\dfrac{\pi}{3} + i\,\mathrm{sen}\,\dfrac{\pi}{3} = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2} = \dfrac{1+i\sqrt{3}}{2}$, so λ is an eingenvalue of $\mathbb{A}$ if and only if $5-4k = -3 \Rightarrow 4k = 8 \Rightarrow k = 2$.

I M 2) $\mathbb{X}$ has coordinates $(2,1,2)$ in the basis $\mathbb{W}$ if $\mathbb{X} = 2\,\mathbb{W}_1 + 1\,\mathbb{W}_2 + 2\,\mathbb{W}_3$.

So we get:

$$\mathbb{W}_3 = \frac{1}{2}\cdot(\mathbb{X} - 2\mathbb{W}_1 - \mathbb{W}_2) = \frac{1}{2}\cdot\left\|\begin{matrix} 3 \\ -2 \\ 1 \end{matrix}\right\| - 1\cdot\left\|\begin{matrix} 1 \\ 1 \\ 0 \end{matrix}\right\| - \frac{1}{2}\cdot\left\|\begin{matrix} 1 \\ 0 \\ -1 \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ -2 \\ 1 \end{matrix}\right\|.$$

I M 3) 1st Method: every eigenvector belongs to $\mathbb{R}^3$ and so the matrix $\mathbb{A}$ is a 3×3 matrix that can be written as $\mathbb{A} = \left\|\begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}\right\|$.

Since $\mathbb{X}_1 = (1,1,0)$ is the eigenvector associated to $\lambda_1 = 0$ we get:

$$\begin{Vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \\ 0 \end{Vmatrix} = 0 \cdot \begin{Vmatrix} 1 \\ 1 \\ 0 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} a+b=0 \\ d+e=0 \\ g+h=0 \end{cases} \Rightarrow \begin{cases} b=-a \\ e=-d \\ h=-g \end{cases}$$

and the matrix $\mathbb{A}$ becomes $\mathbb{A} = \begin{Vmatrix} a & -a & c \\ d & -d & f \\ g & -g & i \end{Vmatrix}$.

Since $\mathbb{X}_2 = (1,0,1)$ is the eigenvector associated to $\lambda_2 = 1$ we get:

$$\begin{Vmatrix} a & -a & c \\ d & -d & f \\ g & -g & i \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 0 \\ 1 \end{Vmatrix} = 1 \cdot \begin{Vmatrix} 1 \\ 0 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 1 \\ 0 \\ 1 \end{Vmatrix} \Rightarrow \begin{cases} a+c=1 \\ d+f=0 \\ g+i=1 \end{cases} \Rightarrow \begin{cases} c=1-a \\ f=-d \\ i=1-g \end{cases}$$

and the matrix $\mathbb{A}$ becomes $\mathbb{A} = \begin{Vmatrix} a & -a & 1-a \\ d & -d & -d \\ g & -g & 1-g \end{Vmatrix}$.

Since $\mathbb{X}_3 = (0,1,1)$ is the eigenvector associated to $\lambda_3 = -1$ we get:

$$\begin{Vmatrix} a & -a & 1-a \\ d & -d & -d \\ g & -g & 1-g \end{Vmatrix} \cdot \begin{Vmatrix} 0 \\ 1 \\ 1 \end{Vmatrix} = -1 \cdot \begin{Vmatrix} 0 \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 0 \\ -1 \\ -1 \end{Vmatrix} \Rightarrow \begin{cases} 1-2a=0 \\ -2d=-1 \\ 1-2g=-1 \end{cases} \Rightarrow \begin{cases} a=\frac{1}{2} \\ d=\frac{1}{2} \\ g=1 \end{cases}$$

and so the matrix $\mathbb{A}$ becomes $\mathbb{A} = \begin{Vmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & 0 \end{Vmatrix}$.

2nd Method: using the similarity relation between matrices, there exists a non singular matrix $\mathbb{P}$ such that $\mathbb{P}^{-1} \cdot \mathbb{A} \cdot \mathbb{P} = \mathbb{D}$ or $\mathbb{A} = \mathbb{P} \cdot \mathbb{D} \cdot \mathbb{P}^{-1}$ where $\mathbb{P} = \| \mathbb{X}_1 \quad \mathbb{X}_2 \quad \mathbb{X}_3 \|$, with $\mathbb{X}_1$, $\mathbb{X}_2$ and $\mathbb{X}_3$ linearly independent eigenvectors of $\mathbb{A}$ and $\mathbb{D}$ is a diagonal 3×3 matrix with elements $(\lambda_1, \lambda_2, \lambda_3) = (0, 1, -1)$.

From $\mathbb{P} = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{Vmatrix}$ we have:

$$\mathbb{P}^{-1} = \frac{1}{|\mathbb{P}|} \left(\text{Adj}(\mathbb{P})\right)^T = \frac{1}{-2} \begin{Vmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{Vmatrix}^T = -\frac{1}{2} \begin{Vmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & -1 \end{Vmatrix} =$$

$= \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Vmatrix}$. So we get:

$$\mathbb{A} = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{Vmatrix} =$$

$$= \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{Vmatrix} = \begin{Vmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -1 & 0 \end{Vmatrix}.$$

I M 4) From the Sylvester Theorem:

$\mathrm{Dim}(\mathrm{Ker}(\mathbb{A})) = \mathrm{Dim}(\mathbb{R}^3) - \mathrm{Dim}(\mathrm{Imm}(\mathbb{A})) = 3 - \mathrm{Rank}(\mathbb{A})$, so $\mathrm{Dim}(\mathrm{Ker}(\mathbb{A}))$ is maximum iff $\mathrm{Rank}(\mathbb{A})$ is minimum.

To calculate $\mathrm{Rank}(\mathbb{A})$ we reduce the matrix using elementary operations on lines:

$$\begin{Vmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & k \\ -1 & m & -1 \end{Vmatrix} \Rightarrow \begin{pmatrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 + R_1 \end{pmatrix} \Rightarrow \begin{Vmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & -3 & k+1 \\ 0 & m+2 & -2 \end{Vmatrix} \Rightarrow$$

$$\Rightarrow \begin{pmatrix} R_3 \leftarrow R_3 - R_2 \\ R_4 \leftarrow R_4 + R_2 \end{pmatrix} \Rightarrow \begin{Vmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & k-1 \\ 0 & m-1 & 0 \end{Vmatrix}.$$

From the last matrix it follows that $\mathrm{Rank}(\mathbb{A}) = \begin{cases} 2 & \text{if } k = m = 1 \\ 3 & \text{otherwise} \end{cases}$; so $\mathrm{Rank}(\mathbb{A})$ is minimum when $k = m = 1$; in such case $\mathbb{A} = \begin{Vmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{Vmatrix}$.

We find the elements of $\mathrm{Ker}(\mathbb{A})$ solving the linear omogeneous system:

$$\begin{Vmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{Vmatrix} 1 & 2 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow$$

$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ -3x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = x_1 + 2x_2 \\ -3x_2 + 2x_1 + 4x_2 = 0 \end{cases} \Rightarrow \begin{cases} x_2 = -2x_1 \\ x_3 = -3x_1 \end{cases}.$$

Every element of $\mathrm{Ker}(\mathbb{A})$ has the form $\begin{Vmatrix} x_1 \\ -2x_1 \\ -3x_1 \end{Vmatrix} = k \cdot \begin{Vmatrix} 1 \\ -2 \\ -3 \end{Vmatrix}$. So a basis for $\mathrm{Ker}(\mathbb{A})$ is $\{(1, -2, -3)\}$.

To find a basis for $\mathrm{Imm}(\mathbb{A})$, we must remember that every element (y_1, y_2, y_3, y_4) of $\mathrm{Imm}(\mathbb{A})$ must satisfy the linear system:

$\begin{Vmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Vmatrix}$. Using the augmented matrix we have:

$\begin{Vmatrix} 1 & 2 & -1 & | & y_1 \\ 2 & 1 & 0 & | & y_2 \\ 1 & -1 & 1 & | & y_3 \\ -1 & 1 & -1 & | & y_4 \end{Vmatrix}$ and using the same elementary operations we get:

$$\begin{Vmatrix} 1 & 2 & -1 & | & y_1 \\ 0 & -3 & 2 & | & y_2 - 2y_1 \\ 0 & -3 & 2 & | & y_3 - y_1 \\ 0 & 3 & -2 & | & y_4 + y_1 \end{Vmatrix} \Rightarrow \begin{Vmatrix} 1 & 2 & -1 & | & y_1 \\ 0 & -3 & 2 & | & y_2 - 2y_1 \\ 0 & 0 & 0 & | & y_3 + y_1 - y_2 \\ 0 & 0 & 0 & | & y_4 - y_1 + y_2 \end{Vmatrix}.$$

Every element of $\mathrm{Imm}(\mathbb{A})$ must satisfy the system:

$\begin{cases} y_3 + y_1 - y_2 = 0 \\ y_4 - y_1 + y_2 = 0 \end{cases} \Rightarrow \begin{cases} y_3 = y_2 - y_1 \\ y_4 = y_1 - y_2 \end{cases}$, so it is a vector like $(y_1, y_2, y_2 - y_1, y_1 - y_2)$.
For $y_1 = 1$ and $y_2 = 0$ we have the vector $\mathbb{Y}_1 = (1, 0, -1, 1)$; for $y_1 = 0$ and $y_2 = 1$ we have the vector $\mathbb{Y}_2 = (0, 1, 1, -1)$; so $\{\mathbb{Y}_1, \mathbb{Y}_2\}$ is a basis for Imm($\mathbb{A}$).

I M 5) An orthogonal matrix that diagonalizes $\mathbb{A}$ is given by:

$\mathbb{U} = \| \mathbb{X}_1 \quad \mathbb{X}_2 \quad \mathbb{X}_3 \quad \mathbb{X}_4 \|$ where $\mathbb{X}_1$, $\mathbb{X}_2$, $\mathbb{X}_3$, $\mathbb{X}_4$ are linear indipendent unit eigenvectors of $\mathbb{A}$.

$$p_{\mathbb{A}}(\lambda) = |\mathbb{A} - \lambda\mathbb{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 1-\lambda \end{vmatrix} =$$
$$= (1-\lambda)\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \\ 1 & 0 & 0 \end{vmatrix} =$$
$$= (1-\lambda)^2\begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} =$$
$$= (1-\lambda)^2\left((1-\lambda)^2 - 1\right) - \left((1-\lambda)^2 - 1\right) = \left((1-\lambda)^2 - 1\right)^2 = \left(\lambda^2 - 2\lambda\right)^2 =$$
$= \lambda^2(\lambda - 2)^2 = 0$. The eigenvalues are $\lambda_1 = \lambda_2 = 0 \;\; \lambda_3 = \lambda_4 = 2$.
The eigenvectors associated to $\lambda = 0$ are the solutions of the omogeneous system:

$$\|\mathbb{A} - 0 \cdot \mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \left\| \begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{matrix} \right\| \cdot \left\| \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\| \Rightarrow \begin{cases} x_1 + x_4 = 0 \\ x_2 + x_3 = 0 \end{cases} \Rightarrow$$
$$\Rightarrow \begin{cases} x_4 = -x_1 \\ x_3 = -x_2 \end{cases}.$$
Every eigenvector associated to $\lambda = 0$ is an eigenvector like $(x_1, x_2, -x_2, -x_1)$.
For $x_1 = 1$ and $x_2 = 0$ we have the eigenvector $(1, 0, 0, -1)$; for $x_1 = 0$ and $x_2 = 1$ we have the eigenvector $(0, 1, -1, 0)$. So we have the two unit eigenvectors $\mathbb{X}_1 = \left(\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}}\right)$ and $\mathbb{X}_2 = \left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$.
The eigenvectors associated to $\lambda = 2$ are the solutions of the omogeneous system:

$$\|\mathbb{A} - 2 \cdot \mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \left\| \begin{matrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{matrix} \right\| \cdot \left\| \begin{matrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{matrix} \right\| = \left\| \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \end{matrix} \right\| \Rightarrow$$
$$\begin{cases} -x_1 + x_4 = 0 \\ -x_2 + x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_4 = x_1 \\ x_3 = x_2 \end{cases}.$$
Every eigenvector associated to $\lambda = 2$ is an eigenvector like (x_1, x_2, x_2, x_1).
For $x_1 = 1$ and $x_2 = 0$ we have the eigenvector $(1, 0, 0, 1)$; for $x_1 = 0$ and $x_2 = 1$ we have the eigenvector $(0, 1, 1, 0)$. So we have the two unit eigenvectors $\mathbb{X}_3 = \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right)$ and $\mathbb{X}_4 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

So an orthogonal matrix that diagonalizes $\mathbb{A}$ is $\mathbb{U} = \left\| \begin{matrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{matrix} \right\|$.