TASK 8/1/2020
QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS

I M 1) If $z = \sqrt{2}\left(2+\sqrt{3}\right)e^{\frac{\pi}{4}i} - 2\left(1+\sqrt{3}\right)e^{\frac{\pi}{3}i}$, calcolate $\sqrt{z}$.

Since $e^{\frac{\pi}{4}i} = \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}$ and

$e^{\frac{\pi}{3}i} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{1}{2} + i\,\frac{\sqrt{3}}{2}$ it is:

$$z = \sqrt{2}\left(2+\sqrt{3}\right)\left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right) - 2\left(1+\sqrt{3}\right)\left(\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right) =$$

$$= \left(2+\sqrt{3}\right)(1+i) - \left(1+\sqrt{3}\right)\left(1+i\sqrt{3}\right) =$$

$$= \left(2+\sqrt{3}-1-\sqrt{3}\right) + i\left(2+\sqrt{3}-\sqrt{3}-3\right) = 1-i.$$

From $1-i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) = \sqrt{2}\cdot\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$ we get:

$$\sqrt{z} = \sqrt{1-i} = \sqrt[4]{2}\cdot\left(\cos\left(\frac{7\pi}{8} + k\,\frac{2\pi}{2}\right) + i\sin\left(\frac{7\pi}{8} + k\,\frac{2\pi}{2}\right)\right), 0 \le k \le 1;$$

And so $c_1 = \sqrt[4]{2}\cdot\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right)$ and $c_2 = \sqrt[4]{2}\cdot\left(\cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}\right)$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & k & 1 \end{Vmatrix}$ determine the two values of the parameter k for which the matrix admits multiple eigenvalues. For these values of k, check if the correspon-ding matrix is diagonalizable or not.

From $|\mathbb{A} - \lambda\,\mathbb{I}| = 0$ we get $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & k & 1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 1 \\ \lambda & k & 1-\lambda \end{vmatrix} =$

$= (-\lambda)\left(\lambda^2 - \lambda - k\right) + \lambda(1+\lambda) = (-\lambda)\left(\lambda^2 - \lambda - k - 1 - \lambda\right) =$

$= (-\lambda)\left(\lambda^2 - 2\lambda - (k+1)\right) = 0$. The first eigenvalue is $\lambda = 0$. To get multiple eigenvalues there are two possibilities.

1) $\lambda^2 - 2\lambda - (k+1) = 0$ for $\lambda = 0 \Rightarrow k+1 = 0 \Rightarrow k = -1$. The characteristic polynomial becomes $(-\lambda)\left(\lambda^2 - 2\lambda\right) = 0 \Rightarrow \lambda^2(\lambda - 2) = 0 \Rightarrow \lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 2$.

For $k = -1$ and $\lambda = 0$ we get:

$\|\mathbb{A} - 0\cdot\mathbb{I}\| = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{Vmatrix}$ and since $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$ we get $\text{Rank}\left(\|\mathbb{A} - 0\cdot\mathbb{I}\|\right) = 2$

and so $m_0^g = 3 - 2 = 1 < m_0^a = 2$ and the matrix, for $k = -1$ and $\lambda = 0$ is not a diagonalizable one.

2) The second degree polynomial $\lambda^2 - 2\lambda - (k+1)$ may have a double root.

Solving we get: $\lambda = 1 \pm \sqrt{1+(k+1)} = 1 \pm \sqrt{k+2}$ and so we get the double root $\lambda = 1$ if $k = -2$. For $k = -2$ and $\lambda = 1$ we get:

$\|\mathbb{A} - 1 \cdot \mathbb{I}\| = \begin{Vmatrix} 0 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & -2 & 0 \end{Vmatrix}$ and since $\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1 \neq 0$ also this time we get:

Rank $(\|\mathbb{A} - 1 \cdot \mathbb{I}\|) = 2$ and so $m_1^g = 3 - 2 = 1 < m_1^a = 2$ and the matrix, for $k = -2$ and $\lambda = 1$ is not a diagonalizable one.

I M 3) Consider the linear map $f : \mathbb{R}^4 \to \mathbb{R}^4$, $\mathbb{Y} = \mathbb{A} \cdot \mathbb{X}$ for which:
$f(x_1, x_2, x_3, x_4) = (x_1 + 3x_2; 2x_1 + 2x_2; x_3 + 2x_4; 2x_3 + 4x_4)$.
Determine the dimensions of the Kernel and of the Image of this linear map, and then find a basis for the Kernel.

From $f(x_1, x_2, x_3, x_4) = (x_1 + 3x_2; 2x_1 + 2x_2; x_3 + 2x_4; 2x_3 + 4x_4)$ we get:

$\mathbb{A} = \begin{Vmatrix} 1 & 3 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{Vmatrix}$. By elementary operations on the rows:

$(R_2 \leftarrow R_2 - 2R_1)$ and $(R_4 \leftarrow R_4 - 2R_3)$ we get:

$\mathbb{A} = \begin{Vmatrix} 1 & 3 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{Vmatrix} \to \begin{Vmatrix} 1 & 3 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{Vmatrix}$. Since $\begin{vmatrix} 1 & 3 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -4 \neq 0$ we see that

Rank $(\mathbb{A}) = 3$ and so $\text{Dim}(\text{Imm}(\mathbb{A})) = 3$ and $\text{Dim}(\text{Ker}(\mathbb{A})) = 4 - 3 = 1$.
To find a basis for the Kernel we must solve the system:

$$\mathbb{A} \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 1 & 3 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x_1 + 3x_2 = 0 \\ 2x_1 + 2x_2 = 0 \\ x_3 + 2x_4 = 0 \\ 2x_3 + 4x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + 3x_2 = 0 \\ -4x_2 = 0 \\ x_3 + 2x_4 = 0 \end{cases} \Rightarrow$$

$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = -2x_4 \end{cases}$. So the vectors belonging to the Kernel are of the form $(0, 0, -2k, k)$ and a basis for the Kernel may be the vector $(0, 0, -2, 1)$.

I M 4) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 2 \\ 2 & 1 \end{Vmatrix}$ determine at least a matrix $\mathbb{B}$, different from $\mathbb{A}$, that is similar to $\mathbb{A}$.

Similarity between matrices is satisfied if it exists a non singular matrix $\mathbb{P}$ such that:
$\mathbb{A} \cdot \mathbb{P} = \mathbb{P} \cdot \mathbb{B} \Rightarrow \mathbb{B} = \mathbb{P}^{-1} \cdot \mathbb{A} \cdot \mathbb{P}$. If we choose $\mathbb{P} = \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix}$ it is $|\mathbb{P}| = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$ and so: $\mathbb{P} = \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix} \Rightarrow (\mathbb{P}^*) = \begin{Vmatrix} 1 & -1 \\ -1 & 2 \end{Vmatrix} \Rightarrow (\mathbb{P}^*)^{\text{T}} = \begin{Vmatrix} 1 & -1 \\ -1 & 2 \end{Vmatrix}$ and finally:

$\mathbb{P}^{-1} = \begin{Vmatrix} 1 & -1 \\ -1 & 2 \end{Vmatrix}$. So, from $\mathbb{B} = \mathbb{P}^{-1} \cdot \mathbb{A} \cdot \mathbb{P}$ we get:

$$\mathbb{B} = \begin{Vmatrix} 1 & -1 \\ -1 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & 2 \\ 2 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 2 & 1 \\ 1 & 1 \end{Vmatrix} = \begin{Vmatrix} 1 & -1 \\ -1 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} 4 & 3 \\ 5 & 3 \end{Vmatrix} = \begin{Vmatrix} -1 & 0 \\ 6 & 3 \end{Vmatrix}.$$

II M 1) Given $f(x,y) = x^2y - xy + xy^2$ and $v = (\cos\alpha, \sin\alpha)$, determine at least two values of α for which it results $D_v f(1,1) = D^2_{v,v} f(1,1)$.

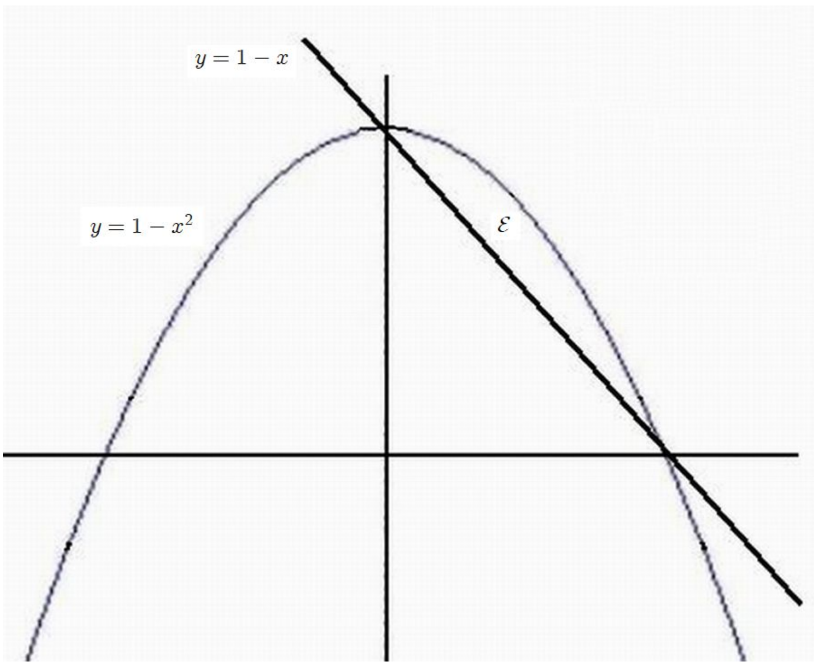
$f(x,y) = x^2y - xy + xy^2$ is a twice differentiable function $\forall (x,y) \in \mathbb{R}^2$.
So $D_v f(1,1) = \nabla f(1,1) \cdot v$ and $D^2_{v,v} f(1,1) = v \cdot \mathbb{H}(1;1) \cdot v^T$.
It is $\nabla f(x,y) = \left(2xy - y + y^2\,; x^2 - x + 2xy\right) \Rightarrow \nabla f(1,1) = (2,2)$.
So we get: $D_v f(1,1) = \nabla f(1,1) \cdot v = (2,2) \cdot (\cos\alpha, \sin\alpha) = 2(\cos\alpha + \sin\alpha)$.
Since $\mathbb{H}(x,y) = \begin{Vmatrix} 2y & 2x - 1 + 2y \\ 2x - 1 + 2y & 2x \end{Vmatrix}$ it is $\mathbb{H}(1,1) = \begin{Vmatrix} 2 & 3 \\ 3 & 2 \end{Vmatrix}$ and so:

$$D^2_{v,v} f(1,1) = \| \cos\alpha \quad \sin\alpha \| \cdot \begin{Vmatrix} 2 & 3 \\ 3 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} \cos\alpha \\ \sin\alpha \end{Vmatrix} = 2\cos^2\alpha + 6\cos\alpha \cdot \sin\alpha + 2\sin^2\alpha .$$

Therefore $D^2_{v,v} f(1,1) = 2 + 6\cos\alpha \cdot \sin\alpha = 2(1 + 3\cos\alpha \cdot \sin\alpha)$.
Then having to be $D_v f(1,1) = D^2_{v,v} f(1,1)$ it will be $\cos\alpha + \sin\alpha = 1 + 3\cos\alpha \cdot \sin\alpha$ and this equality is certainly verified at least for $\alpha = 0$ and for $\alpha = \dfrac{\pi}{2}$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x^2 + y^2 \\ \text{u.c.: } \begin{cases} y \le 1 - x^2 \\ 1 - x \le y \end{cases} \end{cases}$.

We rewrite the problem as $\begin{cases} \text{Max/min } f(x,y) = x^2 + y^2 \\ \text{u.c.: } \begin{cases} x^2 + y - 1 \le 0 \\ 1 - x - y \le 0 \end{cases} \end{cases}$.
The objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a compact set and therefore there are certainly maximum and minimum values.



As can be seen from the figure, it is $f(x,y) \ge 0, \forall (x,y) \in \mathcal{E}$.

Using Kuhn-Tucker conditions, we form the Lagrangian function:

$$\Lambda(x,y,\lambda_1,\lambda_2) = x^2+y^2-\lambda_1\left(x^2+y-1\right)-\lambda_2\left(1-x-y\right).$$

1) case $\lambda_1=0, \lambda_2=0$:

$$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = 2y = 0 \\ x^2+y-1\le 0 \\ 1-x-y\le 0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ 0-0-1\le 0 \\ 1-0-0\le 0: \text{ not satisfied} \end{cases} \quad \text{in fact } (0;0)\notin\mathcal{E}.$$

2) case $\lambda_1\ne 0, \lambda_2=0$:

$$\begin{cases} \Lambda'_x = 2x-2\lambda_1 x = 2x\left(1-\lambda_1\right)=0 \\ \Lambda'_y = 2y-\lambda_1 = 0 \\ y=1-x^2 \\ 1-x\le y \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \\ \lambda_1 = 2>0 \\ 1\le 1 \end{cases}$$

so $(0,1)$ is a possible Maximum point; or:

$$\begin{cases} \Lambda'_x = 2x-2\lambda_1 x = 2x\left(1-\lambda_1\right)=0 \\ \Lambda'_y = 2y-\lambda_1 = 0 \\ y=1-x^2 \\ 1-x\le y \end{cases} \Rightarrow \begin{cases} x^2 = 1-\frac{1}{2}=\frac{1}{2} \\ y=\frac{1}{2} \\ \lambda_1 = 1 \\ 1-x\le y \end{cases} \Rightarrow \begin{cases} x^2 = \frac{1}{2} \\ y=\frac{1}{2} \\ \lambda_1 = 1 \\ 1-x\le y \end{cases}$$

from which:

$$\begin{cases} x=\frac{1}{\sqrt{2}} \\ y=\frac{1}{2} \\ \lambda_1 = 1>0 \\ 1-\frac{1}{\sqrt{2}}\le\frac{1}{2}\Rightarrow\frac{1}{2}\le\frac{1}{\sqrt{2}}: \text{ satisfied} \end{cases} \quad \text{so } \left(\frac{1}{\sqrt{2}},\frac{1}{2}\right) \text{ is a possible Maximum point,}$$

while

$$\begin{cases} x=-\frac{1}{\sqrt{2}} \\ y=\frac{1}{2} \\ \lambda_1 = 1 \\ 1+\frac{1}{\sqrt{2}}\le\frac{1}{2}: \text{ not satisfied} \end{cases}, \text{ in fact } \left(-\frac{1}{\sqrt{2}},\frac{1}{2}\right)\notin\mathcal{E}.$$

3) case $\lambda_1=0, \lambda_2\ne 0$:

$$\begin{cases} \Lambda'_x = 2x+\lambda_2 = 0 \\ \Lambda'_y = 2y+\lambda_2 = 0 \\ y=1-x \\ y\le 1-x^2 \end{cases} \Rightarrow \begin{cases} x=-\frac{\lambda_2}{2} \\ y=-\frac{\lambda_2}{2} \\ -\frac{\lambda_2}{2}=1+\frac{\lambda_2}{2} \\ y\le 1-x^2 \end{cases} \Rightarrow \begin{cases} x=\frac{1}{2} \\ y=\frac{1}{2} \\ \lambda_2 = -1<0 \\ \frac{1}{2}\le 1-\frac{1}{4}: \text{ vera} \end{cases}$$

; from $\lambda_2<0$ it follows that the point $\left(\frac{1}{2},\frac{1}{2}\right)$ is a possible Minimum point.

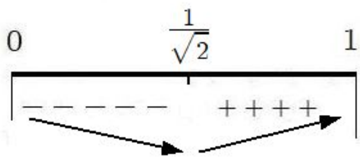
4) case $\lambda_1\ne 0, \lambda_2\ne 0$:

$$\begin{cases} \Lambda'_x = 2x-2\lambda_1 x+\lambda_2 = 0 \\ \Lambda'_y = 2y-\lambda_1+\lambda_2 = 0 \\ y=1-x^2 \\ y=1-x \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \\ \lambda_2 = 0 \\ 2-\lambda_1+\lambda_2=0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \\ \lambda_1 = 2 \\ \lambda_2 = 0 \end{cases} : (0,1) \text{ already studied,}$$

and:

$$\begin{cases} x = 1 \\ y = 0 \\ 2 - 2\lambda_1 + \lambda_2 = 0 \\ -\lambda_1 + \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda_1 = 2 > 0 \\ \lambda_2 = 2 > 0 \end{cases}$$ so $(1,0)$ is a possible Maximum point.

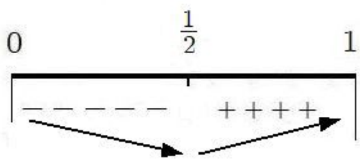
Let's study the objective function $f(x,y) = x^2 + y^2$ on the points of the first constraint $y = 1 - x^2$. Since:
$f(x, 1-x^2) = x^2 + \left(1 - x^2\right)^2 = x^4 - x^2 + 1$ it is $f'(x, 1-x^2) = 4x^3 - 2x$ from this we get $f'(x, 1-x^2) = 2x\left(2x^2 - 1\right) \geq 0$ if $x \geq \dfrac{1}{\sqrt{2}}$ (in $\mathcal{E}$ it is $0 \leq x \leq 1$. Therefore:



and so the point $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$ is, relative to the boundary points only, a Minimum point, contradicting the previous indication $(\lambda_1 = 1 > 0)$ which indicated it as a possible maximum point. So $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$ it is neither a maximum nor a minimum point.
The same conclusion could be reached using the bordered Hessian matrix of the Lagrangian function $\Lambda(x, y, \lambda_1) = x^2 + y^2 - \lambda_1\left(x^2 + y - 1\right)$. It is (for $\lambda_1 = 1$):

$$\overline{\mathbb{H}}(x,y) = \left\| \begin{matrix} 0 & 2x & 1 \\ 2x & 2 - 2\lambda_1 & 0 \\ 1 & 0 & 2 \end{matrix} \right\| \Rightarrow \left| \overline{\mathbb{H}}\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right) \right| = \left| \begin{matrix} 0 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & 0 \\ 1 & 0 & 2 \end{matrix} \right| = -\sqrt{2} \cdot 2\sqrt{2} < 0,$$

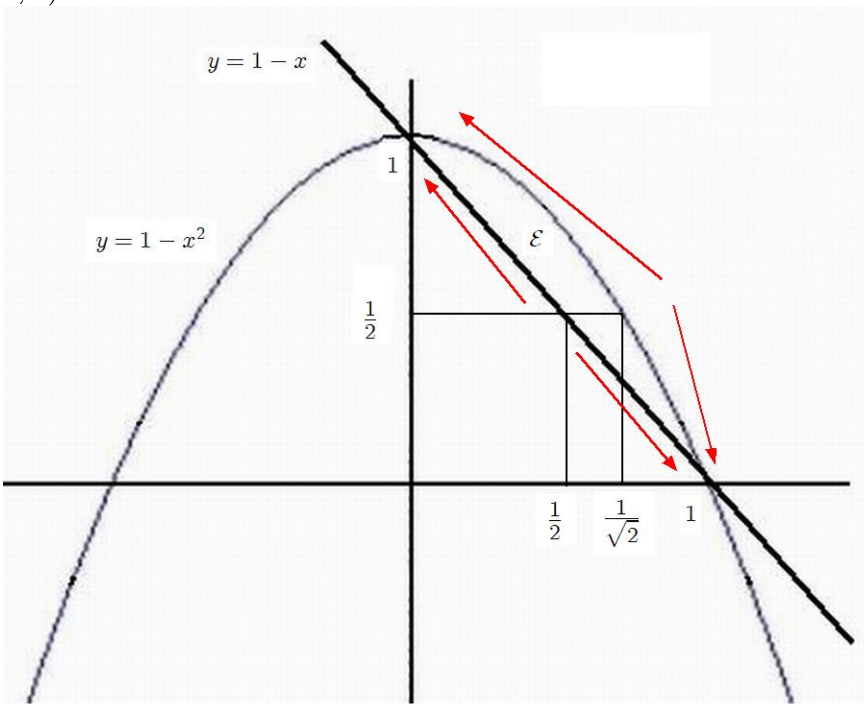
and this result shows us again $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$ as a minimum point.
Now let's study the objective function $f(x,y) = x^2 + y^2$ on the points of the second constraint $y = 1 - x$.
Since $f(x, 1-x) = x^2 + (1-x)^2 = 2x^2 - 2x + 1$ it is $f'(x, 1-x) = 4x - 2$ from which $f'(x, 1-x) = 2(2x - 1) \geq 0$ if $x \geq \dfrac{1}{2}$ (in $\mathcal{E}$ it is $0 \leq x \leq 1$. Therefore:



so we have confirmation that the point $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ is a minimum point, conclusion already ensured by the Weierstrass Theorem, since the point $\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ was the only candidate found for a minimum point. Here too we could confirm using the bordered Hessian matrix of the Lagrangian function $\Lambda(x, y, \lambda_2) = x^2 + y^2 - \lambda_2(1 - x - y)$. It is:

$$\overline{\mathbb{H}}(x,y) = \left\| \begin{matrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{matrix} \right\| \Rightarrow \left| \overline{\mathbb{H}}\left(\frac{1}{2},\frac{1}{2}\right) \right| = \left| \begin{matrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{matrix} \right| = -4 < 0$$, a result that shows us again $\left(\frac{1}{2},\frac{1}{2}\right)$ as a minimum point.

So $\left(\frac{1}{2},\frac{1}{2}\right)$ is the minimum point with $f\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1}{2}$; $(0,1)$ are $(1,0)$ maximum points, with $f(0,1) = f(1,0) = 1$.



II M 3) Given the equation $f(x,y) = x^3y + xy^3 - 2xy - 2x + 2y = 0$, satisfied at $(1,1)$, verify that an implicit function $x \to y(x)$ can be defined with it and that such function has a stationary point. Then determine the nature of this stationary point.

Since $\nabla f(x,y) = \left(3x^2y + y^3 - 2y - 2; x^3 + 3xy^2 - 2x + 2\right)$ it is $\nabla f(1,1) = (0,4)$. It is therefore possible to define an implicit function $x \to y(x)$ for which $y'(1) = -\frac{0}{4} = 0$.

So $x = 1$ is a stationary point for the implicit function.

From $\mathbb{H}(x,y) = \left\| \begin{matrix} 6xy & 3x^2 + 3y^2 - 2 \\ 3x^2 + 3y^2 - 2 & 6xy \end{matrix} \right\|$ we get $\mathbb{H}(1,1) = \left\| \begin{matrix} 6 & 4 \\ 4 & 6 \end{matrix} \right\|$.

From: $y'' = -\dfrac{f''_{xx} + 2f''_{xy}y' + f''_{yy}(y')^2}{f'_y}$, we get $y''(1) = -\dfrac{6 + 8 \cdot 0 + 6 \cdot 0^2}{4} = -\dfrac{3}{2} < 0$.

From $y'(1) = 0$ and $y''(1) = -\frac{3}{2} < 0$ we get that $x = 1$ is a maximum point for the implicit function.

II M 4) Given the vectors $\mathbb{X} = (xy, 2 - 3y)$ and $\mathbb{Y} = (x + 4, xy)$, determine if pairs (x,y) exist for which the scalar product of the two vectors $\mathbb{X} \cdot \mathbb{Y} = f(x,y)$ is maximum or minimum.

$f(x,y) = \mathbb{X} \cdot \mathbb{Y} = (xy, 2 - 3y) \cdot (x + 4, xy) = x^2y + 4xy + 2xy - 3xy^2$.

So $f(x,y) = x^2y + 6xy - 3xy^2$. We apply first order conditions:

$\nabla f(x,y) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 2xy + 6y - 3y^2 = y(2x + 6 - 3y) = 0 \\ f'_y = x^2 + 6x - 6xy = x(x + 6 - 6y) = 0 \end{cases}$ and so we get four possible solutions:

$$\begin{cases} x = 0 \\ y = 0 \end{cases} \cup \begin{cases} x = 0 \\ y = 2 \end{cases} \cup \begin{cases} x = -6 \\ y = 0 \end{cases} \cup \begin{cases} x = 6y - 6 \\ 12y - 12 + 6 - 3y = 0 \end{cases} \Rightarrow \begin{cases} x = -2 \\ y = \frac{2}{3} \end{cases}.$$

For the second order conditions we construct the Hessian matrix:

$$\mathbb{H}(x,y) = \begin{Vmatrix} 2y & 2x + 6 - 6y \\ 2x + 6 - 6y & -6x \end{Vmatrix}.$$

Since $\mathbb{H}(0,0) = \begin{Vmatrix} 0 & 6 \\ 6 & 0 \end{Vmatrix}$ it is $|\mathbb{H}_2| = -36 < 0$ $(0,0)$ is a saddle point;

Since $\mathbb{H}(0,2) = \begin{Vmatrix} 4 & -6 \\ -6 & 0 \end{Vmatrix}$ it is $|\mathbb{H}_2| = -36 < 0$ $(0,2)$ is a saddle point;

Since $\mathbb{H}(-6,0) = \begin{Vmatrix} 0 & -6 \\ -6 & 36 \end{Vmatrix}$ it is $|\mathbb{H}_2| = -36 < 0$ $(-6,0)$ is a saddle point;

Since $\mathbb{H}\left(-2, \dfrac{2}{3}\right) = \begin{Vmatrix} \frac{4}{3} & -2 \\ -2 & 12 \end{Vmatrix}$ it is $\begin{cases} |\mathbb{H}_1| = \frac{4}{3} > 0; |\mathbb{H}_1| = 12 > 0 \\ |\mathbb{H}_2| = 16 - 4 = 12 > 0 \end{cases}$, so $\left(-2, \frac{2}{3}\right)$ is a minimum point.