Compito di Analisi Matematica del 22/9/2017 $\boxed{AM1}$

IM1) $z = i-2$. $\bar{z} = -2-i \Rightarrow w = \dfrac{i-z}{i-\bar{z}} = \dfrac{i-i+2}{i+2+i} = \dfrac{2}{2+2i} = \dfrac{1}{1+i}\cdot\dfrac{1-i}{1-i} = \dfrac{1}{2}(1-i)$.

$\frac{1}{2}(1-i) = \frac{1}{2}\cdot\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) = 2^{-\frac{1}{2}}\cdot\left(\cos\frac{7}{4}\pi + i\,\text{sen}\,\frac{7}{4}\pi\right)$.

$\sqrt[3]{w} = 2^{-\frac{1}{6}}\cdot\left(\cos\left(\frac{7}{12}\pi + K\cdot\frac{2\pi}{3}\right) + i\,\text{sen}\left(\frac{7}{12}\pi + K\frac{2\pi}{3}\right)\right),\quad 0\le K\le 2.$

$\underline{K=0}$: $\frac{1}{\sqrt[6]{2}}\cdot\left(\cos\frac{7}{12}\pi + i\,\text{sen}\,\frac{7}{12}\pi\right)$; $\underline{K=1}$: $\frac{1}{\sqrt[6]{2}}\left(\cos\frac{5}{4}\pi + i\,\text{sen}\,\frac{5}{4}\pi\right)$; $\underline{K=2}$: $\frac{1}{\sqrt[6]{2}}\left(\cos\frac{23}{12}\pi + i\,\text{sen}\,\frac{23}{12}\pi\right)$.

IM2) $f(x;y) = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2} & : (x;y)\ne(0;0) \\ 0 & : (x;y)=(0;0) \end{cases}$.

Continuità in $(0;0)$: $\displaystyle\lim_{(x;y)\to(0;0)} \frac{x^3+y^3}{x^2+y^2} \Rightarrow \lim_{\rho\to0}\frac{\rho^3(\cos^3\vartheta+\text{sen}^3\vartheta)}{\rho^2} = \lim_{\rho\to0}\rho\cdot(\cos^3\vartheta+\text{sen}^3\vartheta) =$

$= 0$. La convergenza è uniforme in quanto $|\rho\cdot(\cos^3\vartheta+\text{sen}^3\vartheta)| < 2\rho$.

$\displaystyle\frac{\partial f}{\partial x}(0;0) = \lim_{h\to0}\frac{1}{h}\cdot\left(\frac{h^3+0}{h^2+0} - 0\right) = \lim_{h\to0}\frac{h^3}{h^3} = 1 = \frac{\partial f}{\partial y}(0;0)$.

Differenziabilità in $(0;0)$: $\displaystyle\lim_{(x;y)\to(0;0)}\frac{1}{\sqrt{x^2+y^2}}\cdot\left(\frac{x^3+y^3}{x^2+y^2} - 0 - (1;1)(x-0;y-0)\right) =$

$\displaystyle = \lim_{(x;y)\to(0;0)}\frac{x^3+y^3-(x+y)\cdot(x^2+y^2)}{\sqrt{x^2+y^2}\cdot(x^2+y^2)} = \lim_{(x;y)\to(0;0)}\frac{-xy^2-x^2y}{\sqrt{x^2+y^2}\cdot(x^2+y^2)} \Rightarrow$

$\displaystyle\Rightarrow \lim_{\rho\to0} -\frac{\rho^3(\cos\vartheta\,\text{sen}^2\vartheta + \cos^2\vartheta\,\text{sen}\,\vartheta)}{\rho\cdot\rho^2} = -\text{sen}\,\vartheta\cos\vartheta(\text{sen}\,\vartheta+\cos\vartheta)\ne0$
eccetto $\vartheta = 0; \frac{\pi}{2}; \pi; \frac{3}{2}\pi; \frac{3}{4}\pi$ e $\frac{7}{4}\pi$.

Quindi la funzione non è differenziabile in $(0;0)$.

IM3) $\begin{cases} f(x;y;z) = 2xy + e^{x-z} + e^{z+y} = 0 \\ g(x;y;z) = 3yz + e^{z-x} + 2e^{z+y} = 0 \end{cases}$; $\begin{cases} f(1;-1;1) = -2+1+1 = 0 \\ g(1;-1;1) = -3+1+2 = 0 \end{cases}$.

$\dfrac{\partial(f;g)}{\partial(x;y;z)} = \left\|\begin{matrix} 2y+e^{x-z} & 2x+e^{z+y} & -e^{x-z}+e^{z+y} \\ -e^{z-x} & 3z+2e^{z+y} & 3y+e^{z-x}+2e^{z+y}\end{matrix}\right\|$; $\dfrac{\partial(f;g)}{\partial(1;-1;1)} = \left\|\begin{matrix} -1 & 3 & 0 \\ -1 & 5 & 0\end{matrix}\right\|$.

Dato che $\begin{vmatrix} -1 & 3 \\ -1 & 5 \end{vmatrix} = -5+3 \neq 0$ si può definire $z \to (x(z); y(z))$

Sarà poi: $\dfrac{dx}{dz} = -\dfrac{\begin{vmatrix} 0 & 3 \\ 0 & 5 \end{vmatrix}}{-2} = \dfrac{dy}{dz} = -\dfrac{\begin{vmatrix} -1 & 0 \\ -1 & 0 \end{vmatrix}}{-2} = 0.$

I M4) $f(x;y) = x^2y - xy^2$. La funzione è due volte differenziabile $\forall (x;y) \in \mathbb{R}^2$.

$\nabla f(x;y) = (2xy - y^2; x^2 - 2xy)$; $\nabla f(1;1) = (2-1; 1-2) = (1;-1)$.

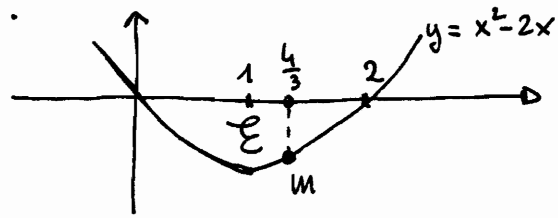
$D_v f(1;1) = (1;-1)\cdot(\cos\alpha; \operatorname{sen}\alpha) = \cos\alpha - \operatorname{sen}\alpha = 0 \Rightarrow \operatorname{sen}\alpha = \cos\alpha \Rightarrow \alpha = \frac{\pi}{4}; \frac{5}{4}\pi.$

$D^2_{v,w} f(1;1) = v \cdot H(1;1) \cdot w^T$. $H(x;y) = \begin{Vmatrix} 2y & 2x-2y \\ 2x-2y & -2x \end{Vmatrix}$; $H(1;1) = \begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix}$.

$D^2_{v,w} f(1;1) = \begin{Vmatrix} \cos\alpha & \operatorname{sen}\alpha \end{Vmatrix} \cdot \begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix} \cdot \begin{Vmatrix} \cos\beta \\ \operatorname{sen}\beta \end{Vmatrix} = (\cos\alpha; \operatorname{sen}\alpha)\cdot(2\cos\beta; -2\operatorname{sen}\beta) =$

$= 2(\cos\alpha\cos\beta - \operatorname{sen}\alpha \operatorname{sen}\beta) = 2\cos(\alpha+\beta).$

II M1) $\begin{cases} \text{Max/min } f(x;y) = xy \\ \text{s.v. } x^2 - 2x \leq y \leq 0 \end{cases}$



La funzione $f(x;y)$ è differenziabile in $\mathbb{R}^2$, la regione E è un insieme limitato e chiuso, i vincoli sono qualificati, per il Teorema di Weierstrass esistono Max e min.

Risulta $f(x;y) \leq 0 \ \forall (x;y) \in E$. Risulta $f(x;y) = 0 \ \forall (x;0)$ con $0 \leq x \leq 2$.

Quindi tutti i punti $(x;0)$ con $0 \leq x \leq 2$ sono punti di Max con $f(x;0) = 0$.

Studiamo il problema sui punti del bordo $y = x^2 - 2x$ in quanto $f(x;y) = x \cdot y$ ha un solo punto stazionario in $(0;0)$ che è già stato studiato.

Da $\Lambda(x;y;\lambda) = xy - \lambda(x^2 - 2x - y)$ si ottiene:

$$\begin{cases} \Lambda'_x = y - 2\lambda x + 2\lambda = 0 \\ \Lambda'_y = x + \lambda = 0 \\ y = x^2 - 2x \end{cases} \Rightarrow \begin{cases} \lambda = -x \\ y + 2x^2 - 2x = 0 \\ y = x^2 - 2x \end{cases} \Rightarrow \begin{cases} \lambda = -x \\ y = 2x - 2x^2 \\ 2x - 2x^2 = x^2 - 2x \end{cases} \Rightarrow \begin{cases} \lambda = -x \\ y = 2x - 2x^2 \\ 3x^2 - 4x = x(3x-4) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda = 0 \end{cases} \text{ già visto } \cup \begin{cases} x = \frac{4}{3} \\ y = -\frac{8}{9} \\ \lambda = -\frac{4}{3} < 0 \end{cases}$$ e quindi $\left(\frac{4}{3}; -\frac{8}{9}\right)$ è il punto di minimo.

AM3

II M 2) $y' = y \cdot \log x \cdot \log y \Rightarrow \dfrac{1}{y \cdot \log y} \cdot y' = \log x$ con $y \neq 0$; $y > 0$; $y \neq 1$, $x > 0$.

$$\int \frac{1}{y \log y} \cdot y' dx = \int \frac{1}{y \log y} dy = \int \frac{1}{\log y} \cdot d(\log y) = \int \frac{1}{t} dt = \log t = \log(\log y);$$

$$\int \log x \, dx = \int 1 \cdot \log x \, dx = x \cdot \log x - \int x \cdot \frac{1}{x} dx = x \cdot \log x - x.$$ Quindi

$$\log(\log y) = x \log x - x + K \Rightarrow \log y = e^{x \log x - x + K} = (e^{\log x})^x \cdot e^{-x} \cdot e^K \Rightarrow$$

$$\Rightarrow \log y = x^x \cdot e^{-x} \cdot m \Rightarrow y = e^{x^x \cdot e^{-x} \cdot m}.$$

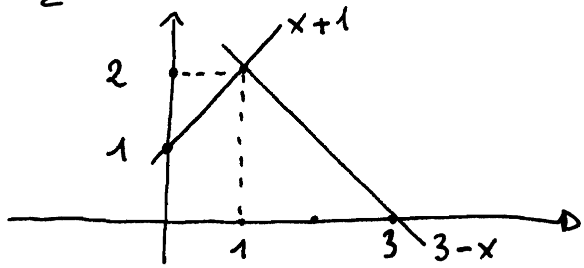
II M 3) $\begin{cases} x' = Ky \\ y' = x + y \end{cases} \Rightarrow \begin{cases} x' - Ky = 0 \\ -x + y' - y = 0 \end{cases} \Rightarrow \begin{Vmatrix} D & -K \\ -1 & D-1 \end{Vmatrix} (x) = 0 \Rightarrow (D^2 - D - K)(x) = 0$

$\Rightarrow \dfrac{1 \pm \sqrt{1+4K}}{2}$. Se una radice è 2 $\Rightarrow \dfrac{1 \pm \sqrt{1+4K}}{2} = 2 \Rightarrow 1 \pm \sqrt{1+4K} = 4 \Rightarrow$

$\Rightarrow \pm\sqrt{1+4K} = 3 \Rightarrow 1 + 4K = 9 \Rightarrow K = 2$. Se $K = 2$ le due soluzioni sono date da 2 e -1. Quindi $x(t) = c_1 e^{2t} + c_2 e^{-t}$. Da $x' = 2y$ si ha:

$$y = \frac{1}{2} x' = \frac{1}{2}\left(2c_1 e^{2t} - c_2 e^{-t}\right) = c_1 e^{2t} - \frac{1}{2} c_2 e^{-t}.$$

II M 4) $\displaystyle\iint_D x^2 + y^2 \, dx\, dy =$



$$= \int_0^1 \int_0^{x+1} x^2 + y^2 \, dy\, dx + \int_1^3 \int_0^{3-x} x^2 + y^2 \, dy\, dx =$$

$$= \int_0^1 \left(x^2 y + \frac{1}{3} y^3 \Big|_0^{x+1} dx + \int_1^3 \left(x^2 y + \frac{1}{3} y^3 \Big|_0^{3-x} dx = \int_0^1 x^2(x+1) + \frac{1}{3}(x+1)^3 dx + \int_1^3 x^2(3-x) + \frac{1}{3}(3-x)^3 dx =$$

$$= \int_0^1 x^3 + x^2 + \frac{1}{3}(x+1)^3 dx + \int_1^3 3x^2 - x^3 + \frac{1}{3}(3-x)^3 dx =$$

$$= \left(\frac{1}{4}x^4 + \frac{x^3}{3} + \frac{1}{12}(x+1)^4 \Big|_0^1 + \left(x^3 - \frac{1}{4}x^4 - \frac{1}{12}(3-x)^4 \Big|_1^3 =$$

$$= \left(\frac{1}{4} + \frac{1}{3} + \frac{16}{12}\right) - \left(0 + 0 + \frac{1}{12}\right) + \left(27 - \frac{81}{4} - 0\right) - \left(1 - \frac{1}{4} - \frac{16}{12}\right) = \frac{23}{12} - \frac{1}{12} + \frac{27}{4} + \frac{7}{12} = \frac{110}{12} = \frac{55}{6}.$$