

$$IM1) z = \frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)} = \frac{1-1+2i-(1-1-2i)}{1-i^2} = \frac{4i}{2} = 2i.$$

$$2i = 2(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) \Rightarrow \sqrt[3]{2i} = \sqrt[3]{2} \left(\cos \left(\frac{\pi}{6} + k \cdot \frac{2\pi}{3} \right) + i \sin \left(\frac{\pi}{6} + k \cdot \frac{2\pi}{3} \right) \right) : 0 \leq k \leq 2.$$

$$\underline{k=0}: \sqrt[3]{2} \cdot \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \sqrt[3]{2} \cdot \left(\frac{\sqrt{3}}{2} + i \frac{1}{2} \right); \underline{k=1}: \sqrt[3]{2} \left(\cos \frac{5}{6}\pi + i \sin \frac{5}{6}\pi \right) = \sqrt[3]{2} \left(-\frac{\sqrt{3}}{2} + i \frac{1}{2} \right);$$

$$\underline{k=2}: \sqrt[3]{2} \left(\cos \frac{3}{2}\pi + i \sin \frac{3}{2}\pi \right) = \sqrt[3]{2} \cdot (-i) = -i\sqrt[3]{2}.$$

$$IM2) \lim_{(x,y) \rightarrow (0,0)} \frac{|xy|^\alpha}{(x^2+y^2)^2} \Rightarrow \lim_{\rho \rightarrow 0} \frac{\rho^{2\alpha} |\cos^\alpha \vartheta \sin^\alpha \vartheta|}{(\rho^2)^2} = \lim_{\rho \rightarrow 0} \rho^{2\alpha-4} \cdot |\cos^\alpha \vartheta \sin^\alpha \vartheta| = 0$$

Se $2\alpha-4 > 0 \Rightarrow \alpha > 2$. Per $\alpha=2$: $\lim_{\rho \rightarrow 0} |\cos^\alpha \vartheta \sin^\alpha \vartheta| = \cos^2 \vartheta \sin^2 \vartheta \neq 0$ e
dispendente da ϑ per cui il limite non esiste.

$$IM3) f(x,y,z) = x^3 + y^3 + z^3 + 3xy + 3yz + 3zx = 1. P=(1;1;-1) : f(P) = 1.$$

$$\nabla f(x,y,z) = (3x^2+3y; 3y^2+3x+3z; 3z^2+3y) : \nabla f(1;1;-1) = (6;3;6).$$

$$\frac{\partial z}{\partial x} = -\frac{6}{6} = -1; \quad \frac{\partial z}{\partial y} = -\frac{3}{6} = -\frac{1}{2}.$$

$$IM4) f(x,y) = \log \left(e^{x^2+y^2} + e^{x^2-y^2} \right). P.E.: \mathbb{R}^2. f \text{ differenziabile.}$$

$$\nabla f(x,y) = \left(\frac{2x e^{x^2+y^2} + 2x e^{x^2-y^2}}{e^{x^2+y^2} + e^{x^2-y^2}}; \frac{2y e^{x^2+y^2} - 2y e^{x^2-y^2}}{e^{x^2+y^2} + e^{x^2-y^2}} \right) = \left(2x; 2y \frac{e^{x^2+y^2} - e^{x^2-y^2}}{e^{x^2+y^2} + e^{x^2-y^2}} \right).$$

$$\nabla f(0,0) = (0,0) \Rightarrow D_y f(0,0) = (0,0) \cdot \left(\frac{\sqrt{2}}{2}; \frac{\sqrt{2}}{2} \right) = 0;$$

$$\nabla f(-1,1) = \left(-2; 2 \cdot \frac{e^2-1}{e^2+1} \right) \Rightarrow D_w f(-1,1) = \left(-2; 2 \frac{e^2-1}{e^2+1} \right) \cdot \left(-\frac{\sqrt{2}}{2}; -\frac{\sqrt{2}}{2} \right) = \frac{2\sqrt{2}}{e^2+1}.$$

$$II M1) \begin{cases} \text{Max/min } f(x,y) = x^4 + y^4 \\ \text{s.t. } x^2 + y^2 \leq 1 \end{cases} \quad f(x,y) \text{ continua, } \Sigma \text{ insieme limitato} \\ \quad \cdot \text{ e chiuso, ma non qualificato.}$$

$$\lambda = x^4 + y^4 - \lambda(x^2 + y^2 - 1).$$

Caso $\lambda = 0$:

$$\begin{cases} \lambda' x = 4x^3 = 0 \\ \lambda' y = 4y^3 = 0 \\ x^2 + y^2 \leq 1 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ 0 \leq 1 \end{cases}; H(x; y) = \begin{vmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{vmatrix} \Rightarrow H(0; 0) = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}; ??$$

Ma $f(x; y) \geq 0 \forall (x; y) \in \mathbb{R}^2$ e $f(0; 0) = 0 \Rightarrow (0; 0)$ punto di minimo.

Caso $\lambda \neq 0$:

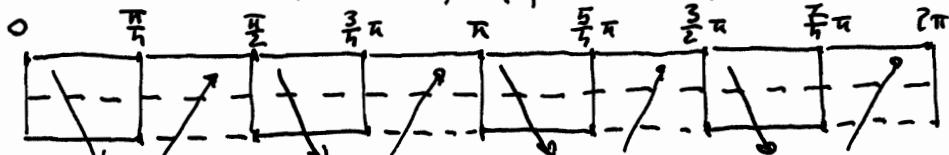
$$\begin{cases} \lambda' x = 4x^3 - 2\lambda x = 2x(2x^2 - \lambda) = 0 \\ \lambda' y = 4y^3 - 2\lambda y = 2y(2y^2 - \lambda) = 0 \\ x^2 + y^2 = 1 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ 0 \neq 1 \end{cases} \text{ già visto} \cup \begin{cases} x=0 \\ \lambda=2 \\ y=\pm 1 \\ (\text{Max?}) \end{cases} \cup \begin{cases} \lambda=2 \\ y=0 \\ x=\pm 1 \\ (\text{Max?}) \end{cases}$$

$$\cup \begin{cases} x^2 = \frac{\lambda}{2} \\ y^2 = \frac{\lambda}{2} \\ \frac{\lambda}{2} + \frac{\lambda}{2} = 1 \end{cases} \Rightarrow \begin{cases} x = \pm \frac{\sqrt{2}}{2} \\ y = \pm \frac{\sqrt{2}}{2} \\ \lambda = 1 \end{cases} (\text{Max?}).$$

Studiamo i punti appartenenti alla frontiera di E : $\begin{cases} x = \cos t \\ y = \sin t \end{cases} \Rightarrow$

$$\Rightarrow f(t) = \cos^4 t + \sin^4 t \Rightarrow f'(t) = 4\cos^3 t (-\sin t) + 4\sin^3 t \cdot \cos t = -4 \sin t \cos t (\cos^2 t - \sin^2 t) = -2 \sin 2t \cos 2t = -\sin 4t. \text{ Da } \sin 4t \geq 0 \text{ per } 0 \leq t \leq \pi \text{ si ottiene:}$$

$$\begin{aligned} \sin 4t \geq 0 \text{ per } (0 \leq t \leq \frac{\pi}{4}) \cup (\frac{\pi}{2} \leq t \leq \frac{3}{4}\pi) \cup (\pi \leq t \leq \frac{5}{4}\pi) \cup (\frac{3}{2}\pi \leq t \leq \frac{7}{4}\pi) \text{ per cos:} \\ f'(t) \geq 0 \text{ per } (\frac{\pi}{4} \leq t \leq \frac{\pi}{2}) \cup (\frac{3}{4}\pi \leq t \leq \pi) \cup (\frac{5}{4}\pi \leq t \leq \frac{3}{2}\pi) \cup (\frac{7}{4}\pi \leq t \leq 2\pi). \end{aligned}$$



Solo i punti $(0; 1); (0; -1); (1; 0)$ e $(-1; 0)$ sono punti di Max.

$$\text{II M2) } \begin{cases} x^1 = x + 2y - 1 \\ y^1 = 2x + y + e^{3t} \end{cases} \Rightarrow \begin{cases} x^1 - x - 2y = -1 \\ -2x + y^1 - y = e^{3t} \end{cases} \Rightarrow \begin{vmatrix} D-1 & -2 \\ -2 & D-1 \end{vmatrix} (x) = \begin{vmatrix} -1 & -2 \\ e^{3t} & D-1 \end{vmatrix} \Rightarrow$$

$$\Rightarrow (D^2 - 2D - 3)(x) = 1 + 2e^{3t} \Rightarrow \lambda^2 - 2\lambda - 3 = 0 \Rightarrow (\lambda - 3)(\lambda + 1) = 0 \Rightarrow \lambda = 3 \text{ e } \lambda = -1.$$

CAM 3

$$\text{Solu\^tione generale omogenea per } x : x(t) = c_1 e^{3t} + c_2 e^{-t}.$$

Trovando $\lambda=3$ una soluzione prenute anche nel termine noto, per trovare le soluzioni particolari dovremo fare $x_0(t) = a \cdot e^{3t} + b t e^{3t} + k$. Solituendo avremo:

$$\text{da } x_0'(t) = 3a e^{3t} + b e^{3t} + 3bt e^{3t} + 0; x_0''(t) = (9a + 6b) e^{3t} + 9bt e^{3t} \Rightarrow \\ 9bt e^{3t} + (9a + 6b) e^{3t} - 2((3a + b) e^{3t} + 3bt e^{3t}) - 3(a e^{3t} + bt e^{3t} + k) = 1 + 2e^{3t} \Rightarrow \\ \Rightarrow (9b - 6b - 3b)t e^{3t} + (9a + 6b - 6a - 2b - 3a)e^{3t} - 3k = 1 + 2e^{3t} \Rightarrow \\ \Rightarrow 4b = 2 \quad e \quad -3k = 1 \Rightarrow b = \frac{1}{2} \quad e \quad k = -\frac{1}{3} \Rightarrow x(t) = c_1 e^{3t} + c_2 e^{-t} + \frac{1}{2} t e^{3t} - \frac{1}{3}.$$

$$\text{Dq } y(t) = \frac{1}{2}(x' - x + 1) \Rightarrow y(t) = c_1 e^{3t} - c_2 e^{-t} + \frac{1}{2} t e^{3t} + \frac{1}{4} e^{3t} + \frac{2}{3}.$$

$$\text{III 3) } \begin{cases} y'(1+x^2) = x(1+y^2) \\ y(0) = 0 \end{cases} \Rightarrow \frac{1}{1+y^2} \cdot y' = \frac{x}{1+x^2} \Rightarrow \int \frac{1}{1+y^2} dy = \int \frac{x}{1+x^2} dx + k \Rightarrow$$

$$\Rightarrow \arctg y = \frac{1}{2} \log(1+x^2) + k \Rightarrow y = \operatorname{tg}\left(\frac{1}{2} \log(1+x^2) + k\right).$$

$$\text{Dq } y(0) = 0 \Rightarrow 0 = \operatorname{tg}(0+k) \Rightarrow k=0. \text{ Soluzione: } y = \operatorname{tg}\left(\frac{1}{2} \log(1+x^2)\right).$$

$$\text{III 4) } \iint_D (x+y) dx dy : D = \{(x,y) \in \mathbb{R}^2 : 1-x \leq y \leq 1-x^2\}$$

$$\iint_D f(x,y) dx dy = \int_0^1 \int_{1-x}^{1-x^2} (x+y) dy dx = \int_0^1 \left(xy + \frac{1}{2} y^2 \right) \Big|_{1-x}^{1-x^2} dx = \\ = \int_0^1 \left(x(1-x^2) + \frac{1}{2}(1-x^2)^2 \right) - \left(x(1-x) + \frac{1}{2}(1-x)^2 \right) dx =$$

$$= \int_0^1 \cancel{x-x^3} + \cancel{\frac{1}{2}x^4} - \cancel{x+x^2} - \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}x^2} + x \ dx = \int_0^1 \frac{1}{2}x^4 - x^3 - \frac{1}{2}x^2 + x \ dx =$$

$$= \left[\frac{1}{10}x^5 - \frac{1}{4}x^4 - \frac{1}{6}x^3 + \frac{1}{2}x^2 \right]_0^1 = \frac{1}{10} - \frac{1}{4} - \frac{1}{6} + \frac{1}{2} = \frac{6-15-10+30}{60} = \frac{11}{60}.$$

