QUANTITATIVE METHODS for ECONOMIC APPLICATIONS

MATHEMATICS for ECONOMIC APPLICATIONS

TASK 3/2/2020

I M 1) Transform $z = \dfrac{1+\sqrt{3}}{1+i} + \dfrac{1-\sqrt{3}}{1-i}$ in trigonometric form and then calculate z^3

$$z = \frac{1+\sqrt{3}}{1+i} + \frac{1-\sqrt{3}}{1-i} = \frac{\left(1+\sqrt{3}\right)(1-i) + \left(1-\sqrt{3}\right)(1+i)}{(1+i)(1-i)} =$$
$$= \frac{\left(1+\sqrt{3}+1-\sqrt{3}\right) + \left(-1-\sqrt{3}+1-\sqrt{3}\right)i}{1+1} = \frac{2-2\sqrt{3}\,i}{2} = 1-\sqrt{3}\,i =$$
$$= 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right) = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right).$$

So $z^3 = 8\left(\cos 5\pi + i\sin 5\pi\right) = 8\left(\cos\pi + i\sin\pi\right) = -8$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 2 & 1 \\ -1 & m & 2 \\ 1 & 1 & k \end{Vmatrix}$ and the vector $\mathbb{X} = \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}$, determine the values of m and k for which the vector $\mathbb{Y} = \mathbb{A}\cdot\mathbb{X}$ is perpendicular to the vector $(0,1,-1)$ and with modulus equal to $\sqrt{24}$.

$$\mathbb{Y} = \mathbb{A}\cdot\mathbb{X} = \begin{Vmatrix} 1 & 2 & 1 \\ -1 & m & 2 \\ 1 & 1 & k \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 4 \\ m+1 \\ k+2 \end{Vmatrix}.$$

$\mathbb{Y} = \mathbb{A}\cdot\mathbb{X}$ perpendicular to $(0,1,-1)$ implies $(4, m+1, k+2)\cdot(0,1,-1) = 0 \Rightarrow$
$\Rightarrow 0 + m + 1 - k - 2 = 0 \Rightarrow m - k = 1 \Rightarrow m = k+1$.

Modulus of $\mathbb{Y}$ equal to $\sqrt{24}$ implies $\|\mathbb{Y}\| = \sqrt{4^2 + (m+1)^2 + (k+2)^2} = \sqrt{24} \Rightarrow$
$\Rightarrow \sqrt{4^2 + (k+2)^2 + (k+2)^2} = \sqrt{4^2 + 2(k+2)^2} = \sqrt{24} \Rightarrow$
$\Rightarrow 2k^2 + 8k + 24 = 24 \Rightarrow 2k(k+4) = 0 \Rightarrow \begin{cases} k = 0 \text{ and } m = k+1 = 1 \\ k = -4 \text{ and } m = k+1 = -3 \end{cases}.$

I M 3) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 0 & 1 \\ 1 & k & 1 \\ 1 & 0 & 1 \end{Vmatrix}$ determine the two values of the parameter k for which the matrix admits multiple eigenvalues. For these values of k, check if the corresponding matrix is diagonalizable or not.

From $|\mathbb{A} - \lambda\,\mathbb{I}| = 0$ we get $\begin{vmatrix} 1-\lambda & 0 & 1 \\ 1 & k-\lambda & 1 \\ 1 & 0 & 1-\lambda \end{vmatrix} = (k-\lambda)\left((1-\lambda)^2 - 1\right) =$
$= (k-\lambda)\left(\lambda^2 - 2\lambda\right) = (k-\lambda)(\lambda)(\lambda-2) = 0 \Rightarrow \lambda_1 = k, \lambda_2 = 0, \lambda_3 = 2$.

To get multiple eigenvalues there are two possibilities.

1) if $k=0 \Rightarrow \lambda_1=\lambda_2=0, \lambda_3=2$ and so we get: $\|\mathbb{A}-0\cdot\mathbb{I}\| = \begin{Vmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{Vmatrix}$;

from $\operatorname{Rank}(\|\mathbb{A}-0\cdot\mathbb{I}\|)=1 \Rightarrow m_0^g = 3-1=2=m_0^a$ and the matrix, for $k=0$ is a diagonalizable one.

2) if $k=2 \Rightarrow \lambda_1=0, \lambda_2=\lambda_3=2$ and so we get: $\|\mathbb{A}-2\cdot\mathbb{I}\| = \begin{Vmatrix} -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{Vmatrix}$; since:

$\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2 \neq 0$, from $\operatorname{Rank}(\|\mathbb{A}-2\cdot\mathbb{I}\|)=2 \Rightarrow m_2^g = 3-2=1 < m_2^a = 2$ and the matrix, for $k=2$ is not a diagonalizable one.

I M 4) Given the basis for $\mathbb{R}^3$: $\mathbb{W} = \{(1,1,0);(1,0,1);(0,1,1)\}$, find the coordinates of the vector $\mathbb{Y}=(0,1,-1)$ in such basis.

To solve the problem we must simply solve the system:

$$\mathbb{Y}=\mathbb{W}\cdot\mathbb{X} = \begin{Vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 1 \\ -1 \end{Vmatrix} \Rightarrow \begin{cases} x_1+x_2=0 \\ x_1+x_3=1 \\ x_2+x_3=-1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x_2=-x_1 \\ x_3=1-x_1 \\ x_2+x_3=-1 \end{cases} \Rightarrow \begin{cases} x_2=-x_1 \\ x_3=1-x_1 \\ -x_1+1-x_1=-1 \end{cases} \Rightarrow \begin{cases} x_1=1 \\ x_2=-1 \\ x_3=0 \end{cases}.$$

II M 1) Given the system $\begin{cases} f(x,y,z) = x\,e^y + y\,e^x - 2\,e^z = 0 \\ g(x,y,z) = x^2y + xz^2 - 2xyz = 0 \end{cases}$ satisfied at $P_0=(1,1,1)$, verify that an implicit function $x \to (y,z)$ can be defined with it, and then calculate the first order derivatives of this function.

The functions $f(x,y,z)$ and $g(x,y,z)$ are differentiable functions $\forall (x,y,z)\in\mathbb{R}^3$.
To apply Dini's Theorem we calculate the Jacobian matrix:

$$\frac{\partial(f,g)}{\partial(x,y,z)} = \begin{Vmatrix} e^y + y\,e^x & xe^y + e^x & -2\,e^z \\ 2xy+z^2-2yz & x^2-2xz & 2xz-2xy \end{Vmatrix}$$

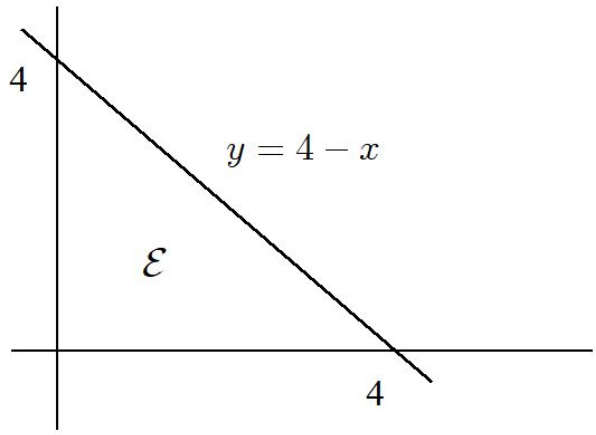
and so $\dfrac{\partial(f,g)}{\partial(x,y,z)}(1,1,1) = \begin{Vmatrix} 2e & 2e & -2e \\ 1 & -1 & 0 \end{Vmatrix}$. Since $\begin{vmatrix} 2e & -2e \\ -1 & 0 \end{vmatrix} = -2e \neq 0$ it is possible to define an implicit function $x \to (y,z)$. For its derivatives we get:

$$\frac{dy}{dx} = -\frac{\begin{vmatrix} 2e & -2e \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 2e & -2e \\ -1 & 0 \end{vmatrix}} = -\frac{2e}{-2e} = 1 \text{ and } \frac{dz}{dx} = -\frac{\begin{vmatrix} 2e & 2e \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 2e & -2e \\ -1 & 0 \end{vmatrix}} = -\frac{4e}{-2e} = 2.$$

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x^2+y^2-x\,y-3x \\ \text{u.c.: } \begin{cases} x \geq 0 \\ y \geq 0 \\ y \leq 4-x \end{cases} \end{cases}$.

The objective function of the problem is a continuous function, the feasible region $\mathcal{E}$ is a compact set and therefore there are certainly maximum and minimum values.

Given the number of constraints, it is not convenient to use the Kuhn-Tucker conditions which would require the resolution of 8 systems.



We firstly study the possible points of free relative maximum and minimum.
Applying the first order conditions we get:

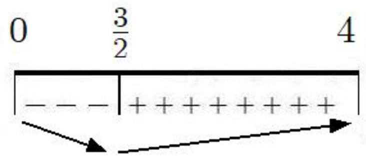
$$\nabla f(x,y) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 2x - y - 3 = 0 \\ f'_y = 2y - x = 0 \end{cases} \Rightarrow \begin{cases} 3y - 3 = 0 \\ x = 2y \end{cases} \Rightarrow \begin{cases} x = 2 \\ y = 1 \end{cases} \text{ and } (2,1) \in \mathcal{E}.$$

For the second order conditions we get, using the Hessian matrix:

$$\mathbb{H}(x,y) = \begin{Vmatrix} 2 & -1 \\ -1 & 2 \end{Vmatrix} = \mathbb{H}(2,1). \text{ Since } \begin{cases} |\mathbb{H}_1| = 2 > 0 \\ |\mathbb{H}_2| = 4 - 1 = 3 > 0 \end{cases}, \text{ the point } (2,1) \text{ is a}$$

minimum point, with $f(2,1) = -3$.

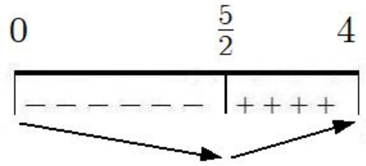
Now we study the objective function $f(x,y) = x^2 + y^2 - x\,y - 3x$ at the points of the first constraint $x = 0$. It is $f(0,y) = y^2$ and for $y > 0$ it is an ever increasing function.
Let's study the objective function $f(x,y) = x^2 + y^2 - x\,y - 3x$ at the points of the second constraint $y = 0$. It is $f(x,0) = x^2 - 3x \Rightarrow f'(x) = 2x - 3 \geq 0$ and therefore the function is increasing for $x \geq \frac{3}{2}$.



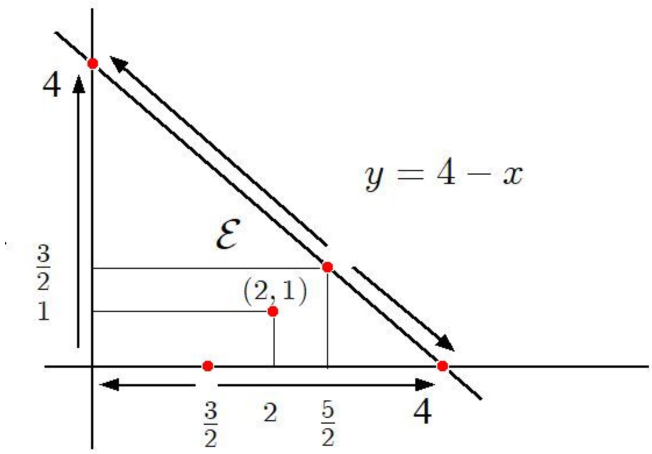
Finally, let's study the objective function $f(x,y) = x^2 + y^2 - x\,y - 3x$ at the points of the third constraint $y = 4 - x$. It is :

$$f(x, 4-x) = x^2 + 16 + x^2 - 8x - 4x + x^2 - 3x = 3x^2 - 15x + 16 \Rightarrow$$

$\Rightarrow f'(x) = 6x - 15 \geq 0$ and therefore the function is increasing for $x \geq \frac{5}{2}$.



We therefore have the following situation:



from which we see that:
- $(0,4)$ is a maximum point, with $f(0,4) = 16$, and it is the absolute maximum;
- $(4,0)$ is a maximum point, with $f(4,0) = 4$, and it is a relative maximum;
- $(2,1)$ is a minimum point, with $f(2,1) = -3$, and it is the absolute minimum.

The point $\left(\frac{3}{2},0\right)$ is a minimum point, with $f\left(\frac{3}{2},0\right) = -\frac{9}{4}$, relatively to the points of the constraint $y = 0$. For a complete analysis we form the Lagrangian function only for this constraint. We get: $\Lambda(x,y,\lambda) = x^2 + y^2 - xy - 3x - \lambda(-y)$. So :

$$\begin{cases} \Lambda'_x = 2x - y - 3 = 0 \\ \Lambda'_y = 2y - x + \lambda = 0 \\ y = 0 \end{cases} \Rightarrow \begin{cases} x = \frac{3}{2} \\ y = 0 \\ \lambda = \frac{3}{2} > 0 \end{cases}$$

and therefore the point $\left(\frac{3}{2},0\right)$ may be, since $\lambda = \frac{3}{2} > 0$, with respect to the interior points of the feasible region $\mathcal{E}$, a maximum point and therefore it is neither a maximum point nor a minimum point.

The point $\left(\frac{5}{2},\frac{3}{2}\right)$ is a minimum point, with $f\left(\frac{5}{2},\frac{3}{2}\right) = -\frac{11}{4}$ relatively to the points of the constraint $y = 4 - x$. For a complete analysis we form the Lagrangian function only for this constraint. We get: $\Lambda(x,y,\lambda) = x^2 + y^2 - xy - 3x - \lambda(x + y - 4)$. So :

$$\begin{cases} \Lambda'_x = 2x - y - 3 - \lambda = 0 \\ \Lambda'_y = 2y - x - \lambda = 0 \\ y = 4 - x \end{cases} \Rightarrow \begin{cases} 6x = 15 \\ y = 4 - x \\ \lambda = 8 - 3x \end{cases} \Rightarrow \begin{cases} x = \frac{5}{2} \\ y = \frac{3}{2} \\ \lambda = \frac{1}{2} > 0 \end{cases}$$

and therefore the point $\left(\frac{5}{2},\frac{3}{2}\right)$ may be, since $\lambda = \frac{1}{2} > 0$, with respect to the interior points of the feasible region $\mathcal{E}$, a maximum point and therefore it is neither a maximum point nor a minimum point.

II M 3) Given $f(x,y) = xy$ and the vectors $\mathbb{V} = (1,1)$ and $\mathbb{W} = (-1,1)$, let v and w be their unit vectors. If $\mathcal{D}_v f(x_0,y_0) = \sqrt{2}$ and $\mathcal{D}_w f(x_0,y_0) = 0$, determine the coordinates of the point (x_0,y_0).

The function $f(x,y) = xy$ it is a polynomial and therefore it is differentiable of any order.
So $D_v f(x_0,y_0) = \nabla f(x_0,y_0) \cdot v$ and $D_w f(x_0,y_0) = \nabla f(x_0,y_0) \cdot w$.
Since $\nabla f(x,y) = (y\,;x) \Rightarrow \nabla f(x_0,y_0) = (y_0,x_0)$.

Now we get $v = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ and $w = \left(-\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$. Then:

$$\begin{cases} \mathcal{D}_v f(x_0, y_0) = \nabla f(x_0, y_0) \cdot v = (y_0, x_0) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2} \\ D_w f(x_0, y_0) = \nabla f(x_0, y_0) \cdot w = (y_0, x_0) \cdot \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} (y_0, x_0) \cdot (1, 1) = 2 \\ (y_0, x_0) \cdot (-1, 1) = 0 \end{cases} \Rightarrow \begin{cases} x_0 + y_0 = 2 \\ x_0 - y_0 = 0 \end{cases} \Rightarrow \begin{cases} x_0 = 1 \\ y_0 = 1 \end{cases}.$$

II M 4) Given the function $f(x, y) = x^2 - xy + y^3$, determine the nature of its stationary points.

Applying the first order conditions we get:

$$\nabla f(x, y) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 2x - y = 0 \\ f'_y = 3y^2 - x = 0 \end{cases} \Rightarrow \begin{cases} y = 2x \\ 12x^2 - x = x(12x - 1) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \cup \begin{cases} x = \frac{1}{12} \\ y = \frac{1}{6} \end{cases}.$$

For the second order conditions we use the Hessian matrix: $\mathbb{H}(x, y) = \begin{Vmatrix} 2 & -1 \\ -1 & 6y \end{Vmatrix}$.

Since $\mathbb{H}(0, 0) = \begin{Vmatrix} 2 & -1 \\ -1 & 0 \end{Vmatrix} \Rightarrow |\mathbb{H}_2| = -1 < 0$ the point $(0, 0)$ is a saddle point;

since $\mathbb{H}\left(\dfrac{1}{12}, \dfrac{1}{6}\right) = \begin{Vmatrix} 2 & -1 \\ -1 & 1 \end{Vmatrix} \Rightarrow \begin{cases} |\mathbb{H}_1| = 2 > 0, |\mathbb{H}_1| = 1 > 0 \\ |\mathbb{H}_2| = 2 - 1 = 1 > 0 \end{cases}$ the point $\left(\frac{1}{12}, \frac{1}{6}\right)$ is a minimum point.