# QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
## TASK 20/4/2020

I M 1) After having determined the complex number $z$ which is the solution of the equation $\frac{z}{1+i} + \frac{z}{1-i} = 2i$, calculate its cubic roots $\sqrt[3]{z}$.

$$\frac{z}{1+i} + \frac{z}{1-i} = z\left(\frac{1}{1+i} + \frac{1}{1-i}\right) = z\left(\frac{1-i+1+i}{(1+i)(1-i)}\right) = z\,\frac{2}{2} = 2i \Rightarrow z = 2i\,.$$

Since $2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ we get :

$$\sqrt[3]{z} = \sqrt[3]{2}\left(\cos\left(\frac{\pi}{6} + k\,\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + k\,\frac{2\pi}{3}\right)\right),\ 0 \le k \le 2\,.$$

For $k = 0$ we get $z_0 = \sqrt[3]{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt[3]{2}\left(\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$ ;

for $k = 1$ we get $z_1 = \sqrt[3]{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \sqrt[3]{2}\left(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right)$ ;

for $k = 2$ we get $z_2 = \sqrt[3]{2}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = \sqrt[3]{2}\,(-i)\,.$

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 1 \\ 0 & k & k-1 \\ 0 & 0 & k^2 \end{Vmatrix}$, since $\lambda = -1$ is an eigenvalue of $\mathbb{A}$, find the value of the real parameter $k$, verify that $\mathbb{A}$ is diagonalizable and then find a matrix that diagonalizes $\mathbb{A}$.

From $|\mathbb{A} - \lambda\,\mathbb{I}| = 0$ we get $\begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & k-\lambda & k-1 \\ 0 & 0 & k^2-\lambda \end{vmatrix} = (1-\lambda)(k-\lambda)\big(k^2-\lambda\big) = 0$

For $\lambda = -1$ we get $(1+1)(k+1)\big(k^2+1\big) = 0 \Rightarrow k+1 = 0 \Rightarrow k = -1\,.$

So the matrix is $\mathbb{A} = \begin{Vmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{Vmatrix}$. We find all the eigenvalues:

$P_3(\lambda) = (1-\lambda)(-1-\lambda)(1-\lambda) = 0 \Rightarrow \lambda_1 = \lambda_2 = 1, \lambda_3 = -1\,.$

For $k = -1$ and $\lambda = 1$ we study the matrix $\|\mathbb{A} - 1\,\mathbb{I}\| = \begin{Vmatrix} 0 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{Vmatrix}$ and since $\text{Rank}(\|\mathbb{A} - 1\,\mathbb{I}\|) = 1 \Rightarrow m_1^g = 3 - 1 = 2 = m_1^a$ and so the matrix is diagonalizable.

Now we go to construct the modal matrix. Firstly we search for two linearly independent eigenvectors corresponding to the eigenvalue $\lambda = 1$. We must solve the system:

$$\|\mathbb{A} - 1\,\mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 0 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} y + z = 0 \\ \forall\, x \end{cases} \Rightarrow \begin{cases} \forall\, x \\ z = -y \end{cases}.$$

So eigenvectors corresponding to the eigenvalue $\lambda = 1$ are $\mathbb{V} = (x, y, -y)\,.$

Two linearly independent eigenvectors may be $\mathbb{V}_1 = (1, 0, 0)$ and $\mathbb{V}_2 = (0, 1, -1)\,.$

Now we search for an eigenvector corresponding to the eigenvalue $\lambda = -1$ solving the sy-stem: $\|\mathbb{A} + 1\,\mathbb{I}\| \cdot \mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 2 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} 2x + y + z = 0 \\ 2z = 0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} y = -2x \\ z = 0 \end{cases}$. Eigenvectors corresponding to $\lambda = -1$ are $\mathbb{V} = (x, -2x, 0)$ and one of them may be $\mathbb{V}_3 = (1, -2, 0)$. Finally we construct the modal matrix:

$\mathbb{P} = \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{Vmatrix}$. And we can see that $\mathbb{A} \cdot \mathbb{P} = \mathbb{P} \cdot \mathbb{D} \Rightarrow$

$$\Rightarrow \begin{Vmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{Vmatrix}.$$

I M 3) Since the linear system $\begin{cases} x_1 + x_2 + x_3 = 0 \\ m x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + k x_3 = 0 \end{cases}$ has the solution $(1, 0, -1)$, calculate the values of the real parameters $m$ and $k$ and then find a basis for the space of the solutions of the system.

Substituting we get: $\begin{cases} 1 + 0 - 1 = 0 \\ m + 0 - 1 = 0 \\ 1 + 0 - k = 0 \end{cases} \Rightarrow \begin{cases} m = 1 \\ k = 1 \end{cases}$ and we get the system:

$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \end{cases} \Rightarrow x_1 + x_2 + x_3 = 0 \Rightarrow x_3 = -x_1 - x_2$.

All the solutions of the system are in the form $(x_1, x_2, -x_1 - x_2)$.

A basis for the space of the solutions of the system may be $\{(1, 0, -1); (0, 1, -1)\}$.

I M 4) Determine the vectors parallel to the vector $\mathbb{X} = (1, -1, 2)$ and with modulus equal to $\sqrt{6}$.

A vector parallel to the vector $\mathbb{X} = (1, -1, 2)$ is a vector of the type:
$\mathbb{Y} = k \cdot (1, -1, 2) = (k, -k, 2k)$. And so we get:
$\|\mathbb{Y}\| = \sqrt{k^2 + k^2 + 4k^2} = \sqrt{6k^2} = \sqrt{6} \Rightarrow k^2 = 1 \Rightarrow k = \pm 1$. So we get two vectors corresponding to the request: $\mathbb{Y}_1 = (1, -1, 2)$ and $\mathbb{Y}_2 = (-1, 1, -2)$.

II M 1) Given the equation $f(x, y) = e^{x^2 + y^2} - e^{x - y} = 0$ satisfied at the point $(0, 0)$, verify that an implicit function $x \to y(x)$ can be defined with it, and then calculate the first order derivative of this function.

The function $f(x, y)$ is a differentiable function $\forall\,(x, y) \in \mathbb{R}^2$. It also turns out:
$\nabla f(x, y) = \left(2x\, e^{x^2 + y^2} - e^{x - y}; 2y\, e^{x^2 + y^2} + e^{x - y}\right)$ for which $\nabla f(0, 0) = (-1; 1)$.
Since $f'_y(0, 0) = 1 \neq 0$ it is possible to define an implicit function $x \to y(x)$.
For its derivative we have: $y'(0) = -\dfrac{-1}{1} = 1$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x - 2y + 3z \\ \text{s.v. } x^2 + y^2 + z^2 = 14 \end{cases}$.

The objective function of the problem is a continuous function, the constraint defines a feasible region $\mathcal{E}$ which is a compact set since it consists of only boundary points (the surface of a sphere) and therefore surely maximum and minimum values exist.
For a problem with equality constraints we construct the Lagrangian function and apply to it the first and second order conditions. We have:
$\Lambda(x,y,z,\lambda) = x - 2y + 3z - \lambda\left(x^2 + y^2 + z^2 - 14\right)$.
Applying the first order conditions we have:

$$\nabla\Lambda(x,y,z,\lambda) = \mathbb{O} \Rightarrow \begin{cases} \Lambda'_x = 1 - 2\lambda x = 0 \\ \Lambda'_y = -2 - 2\lambda y = 0 \\ \Lambda'_z = 3 - 2\lambda z = 0 \\ x^2 + y^2 + z^2 = 14 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = -\frac{1}{\lambda} \\ z = \frac{3}{2\lambda} \\ \frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} = 14 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = -\frac{1}{\lambda} \\ z = \frac{3}{2\lambda} \\ \frac{14}{4\lambda^2} = 14 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = -\frac{1}{\lambda} \\ z = \frac{3}{2\lambda} \\ \lambda^2 = \frac{1}{4} \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -2 \\ z = 3 \\ \lambda = \frac{1}{2} \end{cases} \cup \begin{cases} x = -1 \\ y = 2 \\ z = -3 \\ \lambda = -\frac{1}{2} \end{cases}.$$

We have only two stationary points: $P_1 = (1, -2, 3)$ and $P_2 = (-1, 2, -3)$.
By Weierstrass' theorem, since $f(1, -2, 3) = 14$ and $f(-1, 2, -3) = -14$ it obviously turns out that $P_1$ is the maximum point while $P_2$ is the minimum point.
If we want to apply the second order conditions we have to built the bordered Hessian matrix:

$$\overline{\mathbb{H}}(x,y,z,\lambda) = \begin{Vmatrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & -2\lambda \end{Vmatrix}$$

to calculate then, in the points $P_1$ and $P_2$, the minors $\overline{\mathbb{H}}_3$ and $\overline{\mathbb{H}}_4$.

Since $\overline{\mathbb{H}}(P_1) = \begin{Vmatrix} 0 & 2 & -4 & 6 \\ 2 & -1 & 0 & 0 \\ -4 & 0 & -1 & 0 \\ 6 & 0 & 0 & -1 \end{Vmatrix}$ it is :

$\overline{\mathbb{H}}_3(P_1) = 20 > 0$ and $\overline{\mathbb{H}}_4(P_1) = -56 < 0$ and so $P_1$ is the maximum point;

Since $\overline{\mathbb{H}}(P_2) = \begin{Vmatrix} 0 & -2 & 4 & -6 \\ -2 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ -6 & 0 & 0 & 1 \end{Vmatrix}$ it is :

$\overline{\mathbb{H}}_3(P_2) = -20 < 0$ and $\overline{\mathbb{H}}_4(P_2) = -56 < 0$ and so $P_2$ is the minimum point.

II M 3) Given $f(x,y) = x\,y$ and $g(x,y) = x^2 + y^2$ determine for which values of parameter $\alpha$ it is $\mathcal{D}_v f(1,1) = \mathcal{D}_v g(1,1)$, with $v = (\cos\alpha, \sin\alpha)$.

The functions $f(x,y) = x\,y$ and $g(x,y) = x^2 + y^2$ are polynomials and therefore are differentiable in any order $\forall\,(x,y) \in \mathbb{R}^2$.

So $D_v f(1,1) = \nabla f(1,1) \cdot v$ and $D_v g(1,1) = \nabla g(1,1) \cdot v$. Since:
$\nabla f(x,y) = (y\,; x) \Rightarrow \nabla f(1,1) = (1,1)$ and $\nabla g(x,y) = (2x; 2y) \Rightarrow \nabla g(1,1) = (2;2)$, to get $\mathcal{D}_v f(1,1) = \mathcal{D}_v g(1,1)$ it will be:
$(1,1)(\cos\alpha, \sin\alpha) = (2;2)(\cos\alpha, \sin\alpha) \Rightarrow \cos\alpha + \sin\alpha = 0 \Rightarrow \cos\alpha = -\sin\alpha$ and so:
$\alpha = \frac{3}{4}\pi$ or $\alpha = \frac{7}{4}\pi$.

II M 4) Given the function $f(x,y,z) = x^2yz^3 - \log(y-x) + e^{y-z}$, determine the gradient vector of the function at $P_0 = (1,2,2)$.

It is $\nabla f(x,y,z) = \left(2xyz^3 - \frac{-1}{y-x}; x^2z^3 - \frac{1}{y-x} + e^{y-z}; 3x^2yz^2 - e^{y-z}\right)$ and so:
$\nabla f(1,2,2) = \left(32 - \frac{-1}{2-1}; 8 - \frac{1}{2-1} + e^{2-2}; 24 - e^{2-2}\right) = (33;8;23)$.