MATHEMATICS for ECONOMIC APPLICATIONS

TASK 23/4/2020

I M 1) After having determined the complex number z which is the solution of the equation $\dfrac{z}{1-i}-\dfrac{z}{1+i}=1$, calculate its square roots $\sqrt{z}$.

$$\frac{z}{1-i}-\frac{z}{1+i}=z\left(\frac{1}{1-i}-\frac{1}{1+i}\right)=z\left(\frac{1+i-1+i}{(1-i)(1+i)}\right)=z\,\frac{2i}{2}=1\Rightarrow z=\frac{1}{i}=-i\,.$$

Since $-i=\cos\dfrac{3\pi}{2}+i\sin\dfrac{3\pi}{2}$ we get :

$$\sqrt{z}=\cos\left(\frac{3\pi}{4}+k\,\frac{2\pi}{2}\right)+i\sin\left(\frac{3\pi}{4}+k\,\frac{2\pi}{2}\right),\ 0\le k\le 1\,.$$

For $k=0$ we get $z_0=\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}=-\dfrac{\sqrt{2}}{2}+i\,\dfrac{\sqrt{2}}{2}$;

for $k=1$ we get $z_1=\cos\dfrac{7\pi}{4}+i\sin\dfrac{7\pi}{4}=\dfrac{\sqrt{2}}{2}-i\,\dfrac{\sqrt{2}}{2}$.

I M 2) Given the matrix $\mathbb{A}=\begin{Vmatrix}1&0&0\\1&k-1&0\\1&k&k\end{Vmatrix}$, since $\lambda=1$ is a multiple eigenvalue for $\mathbb{A}$, find the values of the real parameter k and check, for such values, if $\mathbb{A}$ is diagonalizable or not.

From $|\mathbb{A}-\lambda\,\mathbb{I}|=0$ we get:

$$\begin{vmatrix}1-\lambda&0&0\\1&k-1-\lambda&0\\1&k&k-\lambda\end{vmatrix}=(1-\lambda)(k-1-\lambda)(k-\lambda)=0\,.$$

To get $\lambda=1$ as a multiple eigenvalue we have two possibilities:

$k-1=0\Rightarrow k=1$ and $k-1-1=0\Rightarrow k=2$.

For $k=1$ the matrix is $\mathbb{A}=\begin{Vmatrix}1&0&0\\1&0&0\\1&1&1\end{Vmatrix}$ and the matrix $\|\mathbb{A}-1\,\mathbb{I}\|$ is $\begin{Vmatrix}0&0&0\\1&-1&0\\1&1&0\end{Vmatrix}$;

since $\begin{vmatrix}1&-1\\1&1\end{vmatrix}=2\neq 0$ we get $\operatorname{Rank}(\mathbb{A}-1\,\mathbb{I})=2\Rightarrow m_1^g=3-2=1<m_1^a=2$ and so the matrix is not a diagonalizable one.

For $k=2$ the matrix is $\mathbb{A}=\begin{Vmatrix}1&0&0\\1&1&0\\1&2&2\end{Vmatrix}$ and the matrix $\|\mathbb{A}-1\,\mathbb{I}\|$ is $\begin{Vmatrix}0&0&0\\1&0&0\\1&2&1\end{Vmatrix}$;

since $\begin{vmatrix}1&0\\1&2\end{vmatrix}=2\neq 0$ we get $\operatorname{Rank}(\mathbb{A}-1\,\mathbb{I})=2\Rightarrow m_1^g=3-2=1<m_1^a=2$ and so the matrix is not a diagonalizable one.

I M 3) Since the linear system $\begin{cases}kx_1+2x_2-x_3=2\\2x_1-kx_2+x_3=2\\3x_1+x_2-kx_3=3\end{cases}$ has the solution $(1,1,1)$, calculate the value of the real parameter k and then find the number of the solutions of the system.

Substituting the solution we get: $\begin{cases} k+2-1=2 \\ 2-k+1=2 \\ 3+1-k=3 \end{cases} \Rightarrow k=1$ and the system becomes:

$\begin{cases} x_1+2x_2-x_3=2 \\ 2x_1-x_2+x_3=2 \\ 3x_1+x_2-x_3=3 \end{cases}$. But $\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & -1 \end{vmatrix} = 5$.

From Cramer's Theorem, since the determinant of the matrix is different from zero the system has one and only one solution.

I M 4) Determine all the vectors orthogonal to the vector $\mathbb{X}_1=(2,-1,1)$ and to the vector $\mathbb{X}_2=(1,1,1)$.

Two vectors are orthogonal if their scalar product is equal to zero.
So a vector $\mathbb{X}=(x,y,z)$ is orthogonal to the vector $\mathbb{X}_1=(2,-1,1)$ if :
$(x,y,z)\cdot(2,-1,1)=2x-y+z=0 \Rightarrow z=y-2x$. Vector $\mathbb{X}$ becomes $(x,y,y-2x)$.
Now the vector $\mathbb{X}=(x,y,y-2x)$ is orthogonal to the vector $\mathbb{X}_2=(1,1,1)$ if :
$(x,y,y-2x)\cdot(1,1,1)=x+y+y-2x=2y-x=0 \Rightarrow x=2y$.
So all the vectors orthogonal to the vector $\mathbb{X}_1=(2,-1,1)$ and to the vector $\mathbb{X}_2=(1,1,1)$ are the vectors $\mathbb{X}=(2y,y,-3y)=k\,(2,1,-3)$.

II M 1) Given the equation $f(x,y)=e^{x+y}-e^{x-y}=0$ satisfied at the point $(0,0)$, verify that an implicit function $x\to y(x)$ can be defined with it, and then calculate the first order derivative of this function.

The function $f(x,y)$ is a differentiable function $\forall\,(x,y)\in\mathbb{R}^2$. It also turns out :
$\nabla f(x,y)=(e^{x+y}-e^{x-y};e^{x+y}+e^{x-y})$ for which $\nabla f(0,0)=(0;2)$.
Since $f'_y(0,0)=2\neq 0$ it is possible to define an implicit function $x\to y(x)$.
For its derivative we have: $y'(0)=-\dfrac{0}{2}=0$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y)=x-y+z \\ \text{s.v. } x^2+y^2+z^2=3 \end{cases}$.

The objective function of the problem is a continuous function, the constraint defines a feasible region $\mathcal{E}$ which is a compact set since it consists of only boundary points (the surface of a sphere) and therefore surely maximum and minimum values exist.
For a problem with equality constraints we construct the Lagrangian function and apply to it the first and second order conditions. We have:
$\Lambda(x,y,z,\lambda)=x-y+z-\lambda\left(x^2+y^2+z^2-3\right)$.
Applying the first order conditions we have:

$$\nabla\Lambda(x,y,z,\lambda)=\mathbb{O}\Rightarrow\begin{cases} \Lambda'_x=1-2\lambda x=0 \\ \Lambda'_y=-1-2\lambda y=0 \\ \Lambda'_z=1-2\lambda z=0 \\ x^2+y^2+z^2=3 \end{cases}\Rightarrow\begin{cases} x=\dfrac{1}{2\lambda} \\ y=-\dfrac{1}{2\lambda} \\ z=\dfrac{1}{2\lambda} \\ \dfrac{1}{4\lambda^2}+\dfrac{1}{4\lambda^2}+\dfrac{1}{4\lambda^2}=3 \end{cases}\Rightarrow$$

$$\Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = -\frac{1}{2\lambda} \\ z = \frac{1}{2\lambda} \\ \frac{3}{4\lambda^2} = 3 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2\lambda} \\ y = -\frac{1}{2\lambda} \\ z = \frac{1}{2\lambda} \\ \lambda^2 = \frac{1}{4} \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -1 \\ z = 1 \\ \lambda = \frac{1}{2} \end{cases} \cup \begin{cases} x = -1 \\ y = 1 \\ z = -1 \\ \lambda = -\frac{1}{2} \end{cases}.$$

We have only two stationary points: $P_1 = (1, -1, 1)$ and $P_2 = (-1, 1, -1)$.
By Weierstrass' theorem, since $f(1, -1, 1) = 3$ and $f(-1, 1, -1) = -3$ it obviously turns out that P_1 is the maximum point while P_2 is the minimum point.
If we want to apply the second order conditions we have to built the bordered Hessian matrix:

$\overline{\mathbb{H}}(x, y, z, \lambda) = \left\| \begin{matrix} 0 & 2x & 2y & 2z \\ 2x & -2\lambda & 0 & 0 \\ 2y & 0 & -2\lambda & 0 \\ 2z & 0 & 0 & -2\lambda \end{matrix} \right\|$ to calculate then, in the points P_1 and P_2, the minors $\overline{\mathbb{H}}_3$ and $\overline{\mathbb{H}}_4$.

Since $\overline{\mathbb{H}}(P_1) = \left\| \begin{matrix} 0 & 2 & -2 & 2 \\ 2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{matrix} \right\|$ it is :

$\overline{\mathbb{H}}_3(P_1) = 8 > 0$ and $\overline{\mathbb{H}}_4(P_1) = -12 < 0$ and so P_1 is the maximum point;

since $\overline{\mathbb{H}}(P_2) = \left\| \begin{matrix} 0 & -2 & 2 & -2 \\ -2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{matrix} \right\|$ it is :

$\overline{\mathbb{H}}_3(P_2) = -8 < 0$ and $\overline{\mathbb{H}}_4(P_2) = -12 < 0$ and so P_2 is the minimum point.

II M 3) Given $f(x, y) = x\,y$ and $g(x, y) = x - y$ determine for which values of parameter α it is $\mathcal{D}_v f(1, 1) = \mathcal{D}_v g(1, 1)$, with $v = (\cos\alpha, \sin\alpha)$.

The functions $f(x, y) = x\,y$ and $g(x, y) = x - y$ are polynomials and therefore are differentiable in any order $\forall\,(x, y) \in \mathbb{R}^2$.
So $D_v f(1, 1) = \nabla f(1, 1) \cdot v$ and $D_v g(1, 1) = \nabla g(1, 1) \cdot v$. Since:
$\nabla f(x, y) = (y\,; x) \Rightarrow \nabla f(1, 1) = (1, 1)$ and $\nabla g(x, y) = (1; -1) \Rightarrow \nabla g(1, 1) = (1; -1)$, to get $\mathcal{D}_v f(1, 1) = \mathcal{D}_v g(1, 1)$ it will be:
$(1, 1)(\cos\alpha, \sin\alpha) = (1; -1)(\cos\alpha, \sin\alpha) \Rightarrow \cos\alpha + \sin\alpha = \cos\alpha - \sin\alpha \Rightarrow \sin\alpha = 0$ and so $\alpha = 0$ or $\alpha = \pi$.

II M 4) Given the function $f(x, y, z) = x^2 y^3 z + \log(z - x) - e^{y-z}$, determine the gradient vector of the function at $P_0 = (1, 2, 2)$.

It is $\nabla f(x, y, z) = \left(2xy^3 z + \frac{-1}{z - x}; 3x^2 y^2 z - e^{y-z}; x^2 y^3 + \frac{1}{z - x} + e^{y-z} \right)$ and so:
$\nabla f(1, 2, 2) = \left(32 + \frac{-1}{2 - 1}; 24 - e^{2-2}; 8 + 1 + e^{2-2} \right) = (31; 23; 10)$.