MATHEMATICS for ECONOMIC APPLICATIONS

TASK 18/6/2020

I M 1) Calculate the cubic roots of the number $z = \dfrac{\sqrt{2}\, i}{1+i}$.

Since $\sqrt{2}\, i = \sqrt{2}\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$ and $1 + i = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$ we get :

$$z = \frac{\sqrt{2}\, i}{1+i} = \frac{\sqrt{2}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)} = \cos\left(\frac{\pi}{2} - \frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{2} - \frac{\pi}{4}\right) =$$

$z = \cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\, i$. For its cubic roots we get:

$$\sqrt[3]{z} = \cos\left(\frac{\pi}{12} + k\,\frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{12} + k\,\frac{2\pi}{3}\right),\ 0 \le k \le 2 .$$

For $k = 0$ we get $z_0 = \cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}$;

for $k = 1$ we get $z_1 = \cos\dfrac{9\pi}{12} + i\sin\dfrac{9\,\pi}{12} = \cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4} = -\dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2}\, i$;

for $k = 2$ we get $z_2 = \cos\dfrac{17\,\pi}{12} + i\sin\dfrac{17\,\pi}{12}$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 0 & -2 \\ 3 & k & 1 \\ 2 & 0 & -3 \end{Vmatrix}$, determine, on varying the parameter k, its eigenvalues and their multiplicity, then establishing if there are values of k for which the given matrix is diagonalizable.

From $|\mathbb{A} - \lambda\,\mathbb{I}| = 0$ we get:

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 3 & k-\lambda & 1 \\ 2 & 0 & -3-\lambda \end{vmatrix} = (k-\lambda)[(1-\lambda)(-3-\lambda)+4] = (k-\lambda)\left(\lambda^2 + 2\lambda + 1\right) =$$

$= (k-\lambda)(\lambda+1)^2 = 0$ if $\lambda_1 = k$ and $\lambda_2 = \lambda_3 = -1$.

If $k = -1$ we get the eigenvalue $\lambda = -1$ whose algebraic multiplicity is $m^a_{-1} = 3$;

if $k \neq -1$ we get the eigenvalue $\lambda = -1$ whose algebraic multiplicity is $m^a_{-1} = 2$.

For $\lambda = -1, \forall\, k$, the matrix $\|\mathbb{A} - \lambda\,\mathbb{I}\|$ becomes $\|\mathbb{A} - (-1)\,\mathbb{I}\| = \begin{Vmatrix} 2 & 0 & -2 \\ 3 & k+1 & 1 \\ 2 & 0 & -2 \end{Vmatrix}$.

But $\begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} = 8 \neq 0$ and so $\text{Rank}\,(\|\mathbb{A} - (-1)\,\mathbb{I}\|) = 2 \Rightarrow m^g_{-1} = 3 - 2 = 1 < m^a_{-1}$.

So the matrix is not diagonalizable $\forall\, k$.

I M 3) Determine, on varying the parameters m and k, the dimensions of the Image and the Kernel of the linear map $\mathbb{R}^5 \to \mathbb{R}^3$, $f(\mathbb{X}) = \mathbb{A} \cdot \mathbb{X}$, with $\mathbb{A} = \begin{Vmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & m & k \\ 0 & 1 & 0 & 1 & m \end{Vmatrix}$.

By Sylvester Theorem we need to calculate the Rank of the matrix $\mathbb{A}$.

By elementary operations on the rows $(R_2 \leftarrow R_2 - R_1)$ we get :
$$\left\|\begin{vmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & m & k \\ 0 & 1 & 0 & 1 & m \end{vmatrix}\right\| \rightarrow \left\|\begin{vmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & m & k-1 \\ 0 & 1 & 0 & 1 & m \end{vmatrix}\right\|$$
By elementary operations on the rows $(R_3 \leftarrow R_3 - R_2)$ we get :
$$\left\|\begin{vmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & m & k-1 \\ 0 & 1 & 0 & 1 & m \end{vmatrix}\right\| \rightarrow \left\|\begin{vmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & m & k-1 \\ 0 & 0 & 0 & 1-m & m-k+1 \end{vmatrix}\right\|$$
If $m \neq 1, \forall\, k$: $\text{Rank}(\mathbb{A}) = 3 \Rightarrow \text{Dim}(\text{Imm}(f)) = 3$ and $\text{Dim}(\text{Ker}(f)) = 5 - 3 = 2$;
if $m = 1, k \neq 2$: $\text{Rank}(\mathbb{A}) = 3 \Rightarrow \text{Dim}(\text{Imm}(f)) = 3$ and $\text{Dim}(\text{Ker}(f)) = 5 - 3 = 2$;
if $m = 1, k = 2$: $\text{Rank}(\mathbb{A}) = 2 \Rightarrow \text{Dim}(\text{Imm}(f)) = 2$ and $\text{Dim}(\text{Ker}(f)) = 5 - 2 = 3$.

I M 4) Given the vectors $\mathbb{X}_1 = (1, 2, 1)$, $\mathbb{X}_2 = (1, -1, 2)$ and $\mathbb{X}_3 = (2, 2, k)$, determine the value of the parameter k for which the three vectors are linearly dependent.

If the three vectors are linearly dependent they must form a singular matrix, and therefore the determinant of the matrix having the three vectors as its lines must be equal to 0.

So we get $|\mathbb{A}| = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 1 & 2 & k \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & -3 & -2 \\ 0 & 1 & k-2 \end{vmatrix} = 1\,(-3k + 6 + 2) = -3k + 8$.

having used elementary operations on the rows $(R_2 \leftarrow R_2 - 2R_1)$ and $(R_3 \leftarrow R_3 - R_1)$.

If $-3k + 8 = 0 \Rightarrow k = \dfrac{8}{3}$ the three vectors are linearly dependent;

if $-3k + 8 \neq 0 \Rightarrow k \neq \dfrac{8}{3}$ the three vectors are linearly independent.

II M 1) Solve the problem : $\begin{cases} \text{Max/min } f(x,y) = x^2 + y \\ \text{u.c. : } (x-1)^2 - 1 \leq y \leq 1 \end{cases}$.

We write the problem in the form $\begin{cases} \text{Max/min } f(x,y) = x^2 + y \\ \text{s.v.: } \begin{cases} (x-1)^2 - 1 - y \leq 0 \\ y - 1 \leq 0 \end{cases} \end{cases}$. The objective function of the problem is a continuous function, the constraints define a feasible region which is a compact set as it is limited and closed and therefore the maximum and minimum values certainly exist. For a problem with inequality constraints we apply the Kuhn-Tucker conditions, we find the solutions and then we study the objective function on the boundary of $\mathcal{E}$.

To apply the Kuhn-Tucker conditions we construct the Lagrangian function :

$\Lambda(x, y, \lambda_1, \lambda_2) = x^2 + y - \lambda_1\big((x-1)^2 - 1 - y\big) - \lambda_2\,(y - 1)$.

Applying the first order conditions we get:

1) case $\lambda_1 = 0, \lambda_2 = 0$:

$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = 1 \neq 0 \\ (x-1)^2 - 1 - y \leq 0 \\ y - 1 \leq 0 \end{cases}$ so there are no solutions.

2) case $\lambda_1 \neq 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_1(x-1) = 0 \\ \Lambda'_y = 1 + \lambda_1 = 0 \\ y = (x-1)^2 - 1 \\ y \le 1 \end{cases} \Rightarrow \begin{cases} 4x - 2 = 0 \\ \lambda_1 = -1 \\ y = (x-1)^2 - 1 \\ y \le 1 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2} \\ \lambda_1 = -1 < 0 \\ y = -\frac{3}{4} \\ -\frac{3}{4} \le 1 \end{cases}$$

: from $\lambda_1 < 0$ it follows that the point $\left(\frac{1}{2}, -\frac{3}{4}\right)$ it is a possible Minimum point.

3) case $\lambda_1 = 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = 1 - \lambda_2 = 0 \\ y = 1 \\ (x-1)^2 - 1 \le y \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_2 = 1 > 0 \\ 0 \le 1 \end{cases}$$

; from $\lambda_2 > 0$ it follows that the point $(0,1)$ is a possible Maximum point.

4) case $\lambda_1 \neq 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = 2x - 2\lambda_1(x-1) = 0 \\ \Lambda'_y = 1 + \lambda_1 - \lambda_2 = 0 \\ y = (x-1)^2 - 1 \\ y = 1 \end{cases} \Rightarrow \begin{cases} x = 1 + \sqrt{2} \\ y = 1 \\ 2x - 2\lambda_1(x-1) = 0 \\ 1 + \lambda_1 - \lambda_2 = 0 \end{cases} \cup \begin{cases} x = 1 - \sqrt{2} \\ y = 1 \\ 2x - 2\lambda_1(x-1) = 0 \\ 1 + \lambda_1 - \lambda_2 = 0 \end{cases};$$

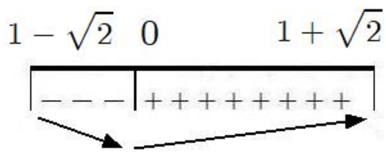
$$\begin{cases} x = 1 + \sqrt{2} \\ y = 1 \\ 2x - 2\lambda_1(x-1) = 0 \\ 1 + \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 + \sqrt{2} \\ y = 1 \\ 2 + 2\sqrt{2} - 2\sqrt{2}\lambda_1 = 0 \\ \lambda_2 = 1 + \lambda_1 \end{cases} \Rightarrow \begin{cases} x = 1 + \sqrt{2} \\ y = 1 \\ \lambda_1 = \frac{1+\sqrt{2}}{\sqrt{2}} > 0 \\ \lambda_2 = \frac{1+2\sqrt{2}}{\sqrt{2}} > 0 \end{cases} :$$

from $\lambda_1 > 0, \lambda_2 > 0$ it follows that the point $\left(1+\sqrt{2}, 1\right)$ is a possible Maximum point;

$$\begin{cases} x = 1 - \sqrt{2} \\ y = 1 \\ 2x - 2\lambda_1(x-1) = 0 \\ 1 + \lambda_1 - \lambda_2 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 - \sqrt{2} \\ y = 1 \\ 2 - 2\sqrt{2} + 2\sqrt{2}\lambda_1 = 0 \\ \lambda_2 = 1 + \lambda_1 \end{cases} \Rightarrow \begin{cases} x = 1 - \sqrt{2} \\ y = 1 \\ \lambda_1 = \frac{\sqrt{2}-1}{\sqrt{2}} > 0 \\ \lambda_2 = \frac{2\sqrt{2}-1}{\sqrt{2}} > 0 \end{cases};$$

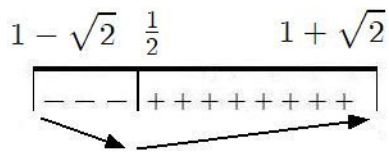
from $\lambda_1 > 0, \lambda_2 > 0$ it follows that the point $\left(1-\sqrt{2}, 1\right)$ is a possible Maximum point.

Now we study the objective function on the constraint $y = 1$.
It is $f(x,1) = x^2 + 1 \Rightarrow f'(x) = 2x \ge 0$ for $x \ge 0$.
Then the function is decreasing for $1 - \sqrt{2} \le x \le 0$ and increasing for $0 \le x \le 1 + \sqrt{2}$.
So it has at $x = 0$ a Minimum point.

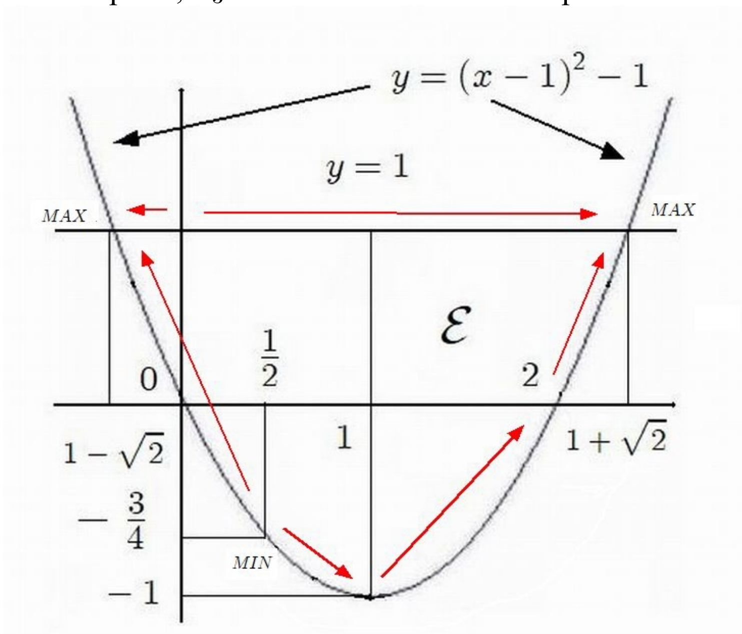


But the point $(0,1)$ had been reported as a possible maximum point and therefore it is neither a maximum point nor a minimum point

We study the objective function on the constraint $y=(x-1)^2-1$.
It is $f\left(x,(x-1)^2-1\right)=2x^2-2x \Rightarrow f'(x)=4x-2\geq 0$ for $x\geq \frac{1}{2}$.
Then the function is decreasing for $1-\sqrt{2}\leq x\leq \frac{1}{2}$ and increasing for $\frac{1}{2}\leq x\leq 1+\sqrt{2}$.
So it has at $x=\frac{1}{2}$ a Minimum point.



The point $P_1=\left(\frac{1}{2},-\frac{3}{4}\right)$ had been reported as a possible minimum point and therefore it is the Minimum point with $f(P_1)=-\frac{1}{2}$;
The point $P_2=\left(1-\sqrt{2},1\right)$, with $f(P_2)=4-2\sqrt{2}$ is a Maximum point;
the point $P_3=\left(1+\sqrt{2},1\right)$, with $f(P_3)=4+2\sqrt{2}$ is a Maximum point;
P_2 it is a relative maximum point, P_3 is the absolute maximum point.



II M 2) Given the function $f(x,y)=e^{x-y}$, find the directions $v=(\cos\alpha,\sin\alpha)$ for which it results $\mathcal{D}_v f(0,0)=0$.

The function $f(x,y)=e^{x-y}$ it is clearly differentiable $\forall\,(x,y)\in\mathbb{R}^2$.
So $D_v f(k,k)=\nabla f(k,k)\cdot v$. Since:
$\nabla f(x,y)=\left(e^{x-y};\,-e^{x-y}\right)\Rightarrow \nabla f(k,k)=(1,\,-1)$, to get $\mathcal{D}_v f(k,k)=0$ it must be :
$(1,\,-1)(\cos\alpha,\sin\alpha)=\cos\alpha-\sin\alpha=0\Rightarrow \cos\alpha=\sin\alpha\Rightarrow \alpha=\frac{\pi}{4}\,;\alpha=\frac{5}{4}\pi$.

II M 3) Given the function $f(x,y,z)=x^2+2y^2+z^2-2xy$, check the nature of its stationary points.

By applying first order conditions we get:

$$\nabla f(x,y,z) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 2x - 2y = 0 \\ f'_y = 4y - 2x = 0 \\ f'_z = 2z = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}. \text{ Then:}$$

$$\mathbb{H}(x,y,z) = \mathbb{H}(0,0,0) = \begin{Vmatrix} 2 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 2 \end{Vmatrix}.$$

Since $\begin{cases} |\mathbb{H}_1| = 2 > 0;\ |\mathbb{H}_1| = 4 > 0 \\ |\mathbb{H}_2| = 8 - 4 > 0;\ |\mathbb{H}_2| = 8 - 0 > 0 \\ |\mathbb{H}_3| = 2\,(8-4) > 0 \end{cases}$ the point $(0,0,0)$ it is a minimum point.

II M 4) Given the equation $f(x,y) = x\,e^{x-y} - y\,e^{y-x} = 0$ satisfied at the point $(0,0)$, verify that an implicit function $x \to y(x)$ can be defined with it, and then calculate the first order derivative of this function.

The function $f(x,y)$ it is a differentiable function $\forall\,(x,y) \in \mathbb{R}^2$. Then it is:
$\nabla f(x,y) = (e^{x-y} + x\,e^{x-y} + y\,e^{y-x};\ -x\,e^{x-y} - e^{y-x} - y\,e^{y-x}) =$
$\nabla f(x,y) = ((1+x)\,e^{x-y} + y\,e^{y-x};\ -x\,e^{x-y} - (1+y)\,e^{y-x}) \Rightarrow \nabla f(0,0) = (1;\ -1)$.
Since $f'_y(0,0) = -1 \neq 0$ it is possible to define an implicit function $x \to y(x)$.
For its derivative we get: $y'(0) = -\dfrac{1}{-1} = 1$.