QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS
TASK 16/7/2020

I M 1) Calculate the square roots of the number $z = \dfrac{1-i}{1+i}$.

Since $1-i = \sqrt{2}\left(\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}\right)$ and $1+i = \sqrt{2}\left(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right)$ we get :

$$z = \frac{1-i}{1+i} = \frac{\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)}{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)} = \cos\left(\frac{7\pi}{4} - \frac{\pi}{4}\right) + i\sin\left(\frac{7\pi}{4} - \frac{\pi}{4}\right) =$$

$z = \cos\dfrac{6\pi}{4} + i\sin\dfrac{6\pi}{4} = \cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2} = -i$. Then we get:

$\sqrt{z} = \cos\left(\dfrac{3\pi}{4} + k\dfrac{2\pi}{2}\right) + i\sin\left(\dfrac{3\pi}{4} + k\dfrac{2\pi}{2}\right),\ 0 \le k \le 1$.

For $k=0$ we get $z_0 = \cos\dfrac{3\pi}{4} + i\sin\dfrac{3\pi}{4} = -\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$;

for $k=1$ we get $z_1 = \cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4} = \dfrac{1}{\sqrt{2}} - \dfrac{1}{\sqrt{2}}i$.

I M 2) Determine an orthogonal matrix that diagonalizes the matrix $\mathbb{A} = \left\|\begin{matrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{matrix}\right\|$.

The matrix is a symmetric one, therefore certainly diagonalizable by means of an orthogonal matrix. From $|\mathbb{A} - \lambda\,\mathbb{I}| = 0$ we get:

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} -\lambda-1 & 1 & 1 \\ 0 & -\lambda & 1 \\ 1+\lambda & 1 & -\lambda \end{vmatrix} = (-\lambda-1)\left(\lambda^2-1\right) + (1+\lambda)(1+\lambda) =$$

$$= (-\lambda-1)\left(\lambda^2 - 1 - (1+\lambda)\right) = -(\lambda+1)\left(\lambda^2-\lambda-2\right) =$$

$$= -(\lambda+1)(\lambda+1)(\lambda-2) = 0$$

and so we get the eigenvalues $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = 2$.

Now we have to find two eigenvectors corresponding to $\lambda = -1$ orthogonal to each other.

So we need to solve the system: $\|\mathbb{A} - (-1)\,\mathbb{I}\| \cdot \mathbb{X} = \mathbb{O}$ that is:

$$\left\|\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\| \Rightarrow x+y+z=0 \Rightarrow z = -x-y.$$

All the eigenvectors corresponding to $\lambda = -1$ are expressible as $(x, y, -x-y)$.

For $x=1$ and $y=0$ we get the first eigenvector: $\mathbb{X}_1 = (1, 0, -1)$.

To obtain the second eigenvector we impose the orthogonality condition :

$\mathbb{X}_1 \cdot \mathbb{X}_2 = (1,0,-1)\cdot(x,y,-x-y) = x+x+y = 0 \Rightarrow y = -2x$.

And so the eigenvectors $(x, -2x, x)$. For $x=1$ we get $\mathbb{X}_2 = (1, -2, 1)$.

Now we have to find one eigenvector corresponding to $\lambda = 2$ and so we need to solve the system: $\|\mathbb{A} - 2\,\mathbb{I}\| \cdot \mathbb{X} = \mathbb{O}$ that is:

$$\left\|\begin{matrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\| \Rightarrow \begin{cases} -2x+y+z=0 \\ x-2y+z=0 \\ x+y-2z=0 \end{cases} \Rightarrow \begin{cases} z=2x-y \\ 3x-3y=0 \Rightarrow \\ 3y-3x=0 \end{cases}$$

$\Rightarrow \begin{cases} y=x \\ z=x \end{cases}$ and so the eigenvectors $\mathbb{X}=(x,x,x)$. For $x=1$ we get $\mathbb{X}_3=(1,1,1)$.

To get an orthogonal matrix we must get the corresponding unit vectors and so :

from $\mathbb{X}_1=(1,0,-1)$ we get $\mathbb{V}_1=\left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right)$;

from $\mathbb{X}_2=(1,-2,1)$ we get $\mathbb{V}_2=\left(\frac{1}{\sqrt{6}},-\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}\right)$;

from $\mathbb{X}_3=(1,1,1)$ we get $\mathbb{V}_3=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

So an orthogonal matrix that diagonalizes the matrix $\mathbb{A}$ is $\mathbb{U}=\left\|\begin{matrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{matrix}\right\|$.

I M 3) Determine, on varying the parameter m, the dimensions of the Image and the Kernel of the linear map $\mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbb{X})=\mathbb{A}\cdot\mathbb{X}$, with $\mathbb{A}=\left\|\begin{matrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 1 & 2 & m \end{matrix}\right\|$.

By Sylvester Theorem we need to calculate the Rank of the matrix $\mathbb{A}$.
By elementary operations on the rows $(R_2 \leftarrow R_2 - 2R_1)$ we get :

$$\left\|\begin{matrix} 1 & 1 & 2 \\ 2 & -1 & 2 \\ 1 & 2 & m \end{matrix}\right\| \to \left\|\begin{matrix} 1 & 1 & 2 \\ 0 & -3 & -2 \\ 1 & 2 & m \end{matrix}\right\|$$

By elementary operations on the rows $(R_3 \leftarrow R_3 - R_2)$ we get :

$$\left\|\begin{matrix} 1 & 1 & 2 \\ 0 & -3 & -2 \\ 1 & 2 & m \end{matrix}\right\| \to \left\|\begin{matrix} 1 & 1 & 2 \\ 0 & -3 & -2 \\ 0 & 1 & m-2 \end{matrix}\right\|.$$

The determinant of the matrix is $\begin{vmatrix} 1 & 1 & 2 \\ 0 & -3 & -2 \\ 0 & 1 & m-2 \end{vmatrix} = 1\cdot(-3m+6+2) = -3m+8$.

If $-3m+8=0 \Rightarrow m=-\frac{8}{3} \Rightarrow$ Rank$(\mathbb{A})=2 \Rightarrow$

$\Rightarrow$ Dim(Imm(f)) $=2$ and Dim(Ker(f)) $=3-2=1$;

if $-3m+8\neq 0 \Rightarrow m\neq -\frac{8}{3} \Rightarrow$ Rank$(\mathbb{A})=3 \Rightarrow$

$\Rightarrow$ Dim(Imm(f)) $=3$ and Dim(Ker(f)) $=3-3=0$.

I M 4) Determine the value of the parameter k for which the vector $\mathbb{X}=(1,k,1)$ can be expressed as a linear combination of the vectors $\mathbb{X}_1=(1,2,-1)$ and $\mathbb{X}_2=(1,1,2)$ and then find the coefficients of such combination.

If $\mathbb{X}$ can be expressed as a linear combination of the other two vectors it means that the three vectors are linearly dependent. Therefore the determinant of the matrix having the three vectors as its lines must be equal to 0:

We get $|\mathbb{A}| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & k \\ -1 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & k-2 \\ 0 & 3 & 2 \end{vmatrix} = 1\,(-2-3k+6) = -3k+4\,.$

If $-3k+4=0 \Rightarrow k=\dfrac{4}{3}$ the three vectors are linearly dependent and so the vector $\mathbb{X} = \left(1, \dfrac{4}{3}, 1\right)$ can be expressed as a linear combination of the vectors $\mathbb{X}_1 = (1, 2, -1)$ and $\mathbb{X}_2 = (1, 1, 2)$. From $\mathbb{X} = \alpha\,\mathbb{X}_1 + \beta\,\mathbb{X}_2$ we get the system:

$$\alpha \left\| \begin{matrix} 1 \\ 2 \\ -1 \end{matrix} \right\| + \beta \left\| \begin{matrix} 1 \\ 1 \\ 2 \end{matrix} \right\| = \left\| \begin{matrix} 1 \\ \frac{4}{3} \\ 1 \end{matrix} \right\| \Rightarrow \begin{cases} \alpha+\beta=1 \\ 2\alpha+\beta=\frac{4}{3} \\ -\alpha+2\beta=1 \end{cases} \Rightarrow \begin{cases} \beta=1-\alpha \\ 2\alpha+1-\alpha=\frac{4}{3} \\ -\alpha+2(1-\alpha)=1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \beta=1-\alpha=\frac{2}{3} \\ \alpha=\frac{1}{3} \\ -\frac{1}{3}+2\left(1-\frac{1}{3}\right)=1 \end{cases} \Rightarrow \begin{cases} \alpha=\frac{1}{3} \\ \beta=\frac{2}{3} \end{cases} \text{ and so } \left(1,\tfrac{4}{3},1\right) = \tfrac{1}{3}(1,2,-1) + \tfrac{2}{3}(1,1,2)\,.$$

II M 1) Solve the problem : Solve the problem : $\begin{cases} \text{Max/min } f(x,y) = x^2 + x\,y^2 \\ \text{u.c.: } 4x^2+y^2 \le 4 \end{cases}$.

The objective function of the problem is a continuous function, the constraint defines a feasible region (ellipse) which is a compact set as it is limited and closed and therefore the maximum and minimum values certainly exist. Being a problem with an inequality constraint we apply the Kuhn-Tucker conditions, find the solutions and then we study the objective function on the frontier of $\mathcal{E}$.

To apply the Kuhn-Tucker conditions we construct the Lagrangian function :
$\Lambda(x,y,\lambda) = x^2 + x\,y^2 - \lambda\left(4x^2+y^2-4\right)$.

Applying the first order conditions we get:

1) case $\lambda = 0$:

$$\begin{cases} \Lambda'_x = 2x + y^2 = 0 \\ \Lambda'_y = 2xy = 0 \\ 4x^2+y^2 \le 4 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=0 \\ 0 \le 4 \end{cases};\ \mathbb{H}(x,y) = \left\| \begin{matrix} 2 & 2y \\ 2y & 2x \end{matrix} \right\| \Rightarrow \mathbb{H}(0,0) = \left\| \begin{matrix} 2 & 0 \\ 0 & 0 \end{matrix} \right\|$$

We can for now only say that $(0,0)$ it is not a Maximum point.

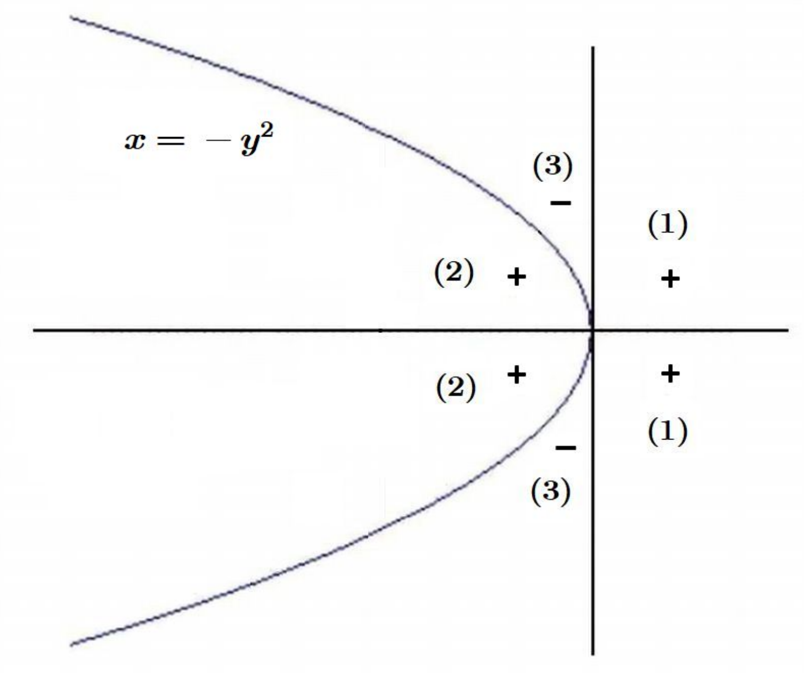
Since $f(0,0)=0$ let's study the sign of the function in a neighborhood of $(0,0)$.

Since $f(x,y) = x\left(x+y^2\right)$ we get:

$f(x,y)>0$ for $\begin{cases} x>0 \\ x>-y^2 \end{cases}$ (1) or for $\begin{cases} x<0 \\ x<-y^2 \end{cases}$ (2) while instead it is:

$f(x,y)<0$ for $\begin{cases} x>0 \\ x<-y^2 \end{cases}$ (impossible) or for $\begin{cases} x<0 \\ x>-y^2 \end{cases}$ (3).

Graphically we have:



So, as we see from the figure, in every neighborhood of $(0,0)$ there are points where $f(x,y) > 0$ and points where $f(x,y) < 0$. The point $(0,0)$ is therefore a saddle point.

2) case $\lambda \neq 0$:

$$\begin{cases} \Lambda'_x = 2x + y^2 - 8\lambda x = 0 \\ \Lambda'_y = 2xy - 2\lambda y = 2y(x-\lambda) = 0 \\ 4x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} 2x(1-4\lambda) = 0 \\ y = 0 \\ x^2 = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda = \frac{1}{4} \end{cases} \Rightarrow \begin{cases} x = -1 \\ y = 0 \\ \lambda = \frac{1}{4} \end{cases};$$

since in both cases it turns out $\lambda = \frac{1}{4} > 0$ these could be Maximum points; or:

$$\begin{cases} 2x + y^2 - 8\lambda x = 0 \\ 2y(x-\lambda) = 0 \\ 4x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} 2x + y^2 - 8x^2 = 0 \\ x = \lambda \\ 4x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} x = \lambda \\ y^2 = 8x^2 - 2x \\ 4x^2 + 8x^2 - 2x = 4 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = \lambda \\ y^2 = 8x^2 - 2x \\ 12x^2 - 2x - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = \lambda \\ y^2 = 8x^2 - 2x \\ 6x^2 - x - 2 = 0 \end{cases}.$$

From $x = \frac{1 \pm \sqrt{1+48}}{12} = \frac{1 \pm 7}{12}$ we get $x = \frac{2}{3}$ or $x = -\frac{1}{2}$ and so:

$$\begin{cases} x = \frac{2}{3} \\ y^2 = 8\,\frac{4}{9} - 2\,\frac{2}{3} \\ \lambda = \frac{2}{3} \end{cases} \Rightarrow \begin{cases} x = \frac{2}{3} \\ y^2 = \frac{20}{9} \\ \lambda = \frac{2}{3} \end{cases} \Rightarrow \begin{cases} x = \frac{2}{3} \\ y = \frac{\sqrt{20}}{3} \\ \lambda = \frac{2}{3} \end{cases} \cup \begin{cases} x = \frac{2}{3} \\ y = -\frac{\sqrt{20}}{3} \\ \lambda = \frac{2}{3} \end{cases}$$

since in both cases it turns out $\lambda = \frac{2}{3} > 0$ these could be Maximum points; or:

$$\begin{cases} x = -\frac{1}{2} \\ y^2 = 8\,\frac{1}{4} - 2\left(-\frac{1}{2}\right) \\ \lambda = -\frac{1}{2} \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2} \\ y^2 = 3 \\ \lambda = -\frac{1}{2} \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{2} \\ y = \sqrt{3} \\ \lambda = -\frac{1}{2} \end{cases} \cup \begin{cases} x = -\frac{1}{2} \\ y = -\sqrt{3} \\ \lambda = -\frac{1}{2} \end{cases}$$

and since in both cases it turns out $\lambda = -\frac{1}{2} < 0$ these could be Minimum points.

Now we study the objective function on the constraint $4x^2 + y^2 = 4$ using the transformation in polar coordinates: $\begin{cases} x = \cos t \\ y = 2\,\text{sen}\, t \end{cases}$. So it is: $f(t) = \cos^2 t + \cos t \cdot 4\,\text{sen}^2 t \Rightarrow$

$\Rightarrow f'(t) = 2\cos t\,(-\,\text{sen}\, t) + (-\,\text{sen}\, t)\left(4\,\text{sen}^2 t\right) + \cos t \cdot 8\,\text{sen}\, t \cos t \Rightarrow$

$\Rightarrow f'(t) = -2\,\text{sen}\, t \cos t - 4\,\text{sen}^3 t + 8\,\text{sen}\, t \cos^2 t \Rightarrow$

$\Rightarrow f'(t) = 2\,\text{sen}\, t\left(-\cos t - 2\,\text{sen}^2 t + 4\cos^2 t\right) \Rightarrow$

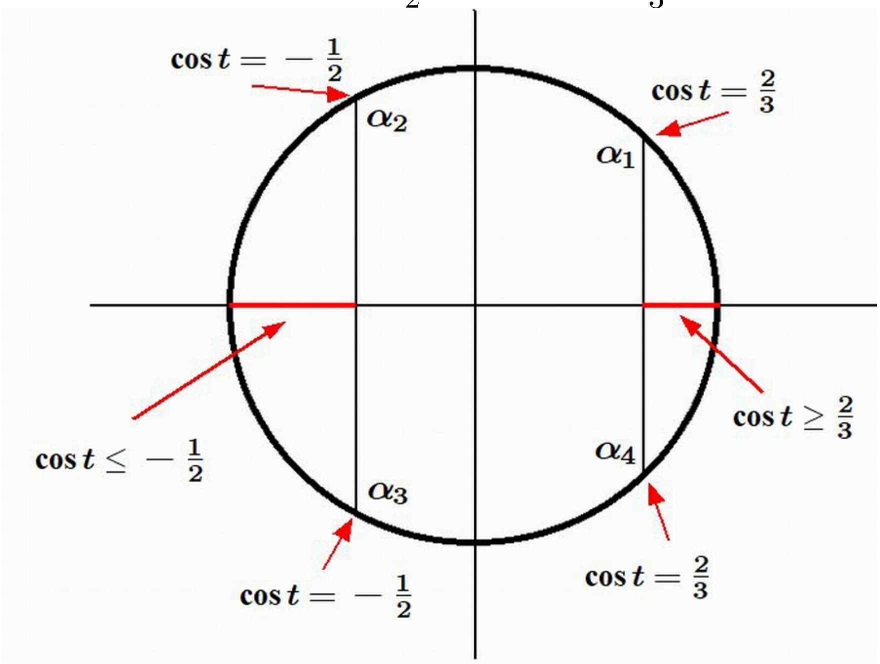
$\Rightarrow f'(t) = 2\,\text{sen}\, t\left(-\cos t - 2\left(1 - \cos^2 t\right) + 4\cos^2 t\right) = 2\,\text{sen}\, t\left(6\cos^2 t - \cos t - 2\right).$

So $f'(t) \geq 0$ for $\begin{cases} \text{sen}\, t \geq 0 \\ 6\cos^2 t - \cos t - 2 \geq 0 \end{cases}$ or $\begin{cases} \text{sen}\, t \leq 0 \\ 6\cos^2 t - \cos t - 2 \leq 0 \end{cases}$.

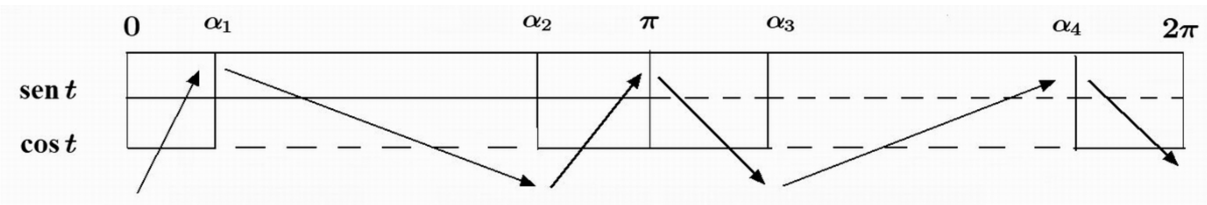
It is $\text{sen}\, t \geq 0$ for $0 \leq t \leq \pi$ and $\text{sen}\, t \leq 0$ for $\pi \leq t \leq 2\pi$.

We get $6\cos^2 t - \cos t - 2 = 0$ for $\cos t = \dfrac{1 \pm \sqrt{1+48}}{12} \Rightarrow \cos t = \dfrac{2}{3}$ and $\cos t = -\dfrac{1}{2}$.

So $6\cos^2 t - \cos t - 2 \geq 0$ for $\cos t \leq -\dfrac{1}{2}$ and for $\cos t \geq \dfrac{2}{3}$.



and therefore we get:



The function $f(t)$ on the boundary of $\mathcal{E}$:
is increasing for $0 \leq t \leq \alpha_1$; is decreasing for $\alpha_1 \leq t \leq \alpha_2$; is increasing for $\alpha_2 \leq t \leq \pi$; is decreasing for $\pi \leq t \leq \alpha_3$; is increasing for $\alpha_3 \leq t \leq \alpha_4$; is decreasing for $\alpha_4 \leq t \leq 2\pi$.
And therefore, relatively to the points of the boundary of $\mathcal{E}$:

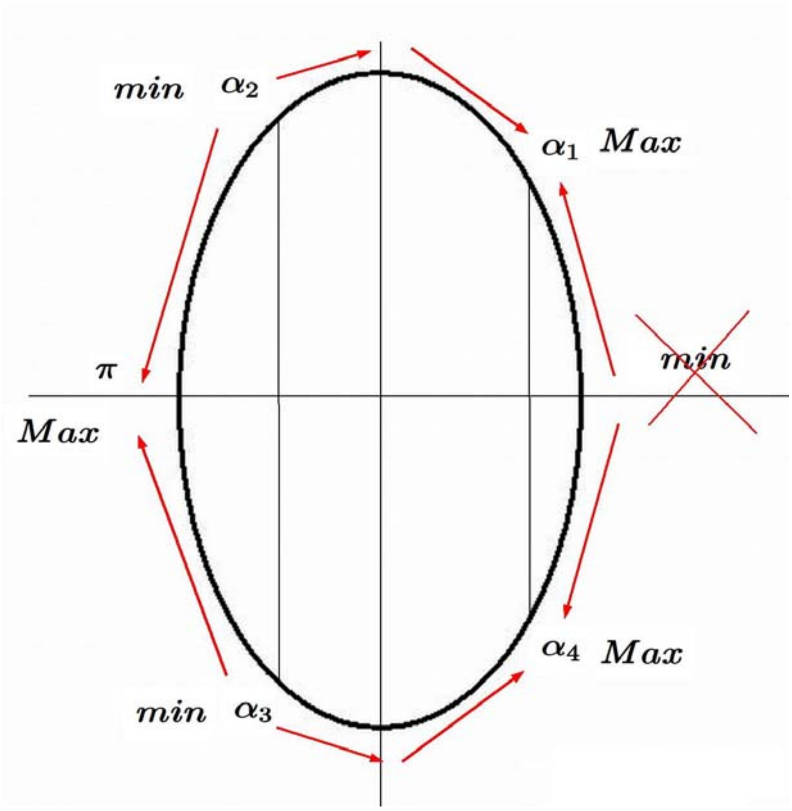
$\left(\frac{2}{3};\frac{\sqrt{20}}{3}\right)$ is a Maximum point, $\left(-\frac{1}{2};\sqrt{3}\right)$ is a Minimum point, $(-1;0)$ is a Maximum point, $\left(-\frac{1}{2};-\sqrt{3}\right)$ is a Minimum point, $\left(\frac{2}{3};-\frac{\sqrt{20}}{3}\right)$ is a Maximum point, $(1;0)$ is a Minimum point.

All the conclusions obtained by studying the boundary coincide with those hypothesized by the sign of the multiplier, except for the point $(1,0)$ that for the analysis on the boundary it seems a minimum point while for the sign of the multiplier it seems to be a maximum point and therefore it is nothing.

Since $f(-1,0)=1$ and $f\left(\frac{2}{3}:\frac{\sqrt{20}}{3}\right)=f\left(\frac{2}{3}:-\frac{\sqrt{20}}{3}\right)=\frac{52}{27}$ the points $\left(\frac{2}{3}:\frac{\sqrt{20}}{3}\right)$ and $\left(\frac{2}{3}:-\frac{\sqrt{20}}{3}\right)$ are absolute maximum points, the point $(-1,0)$ is a relative maximum point.

The absolute minimum value is obtained with $f\left(-\frac{1}{2}:\sqrt{3}\right)=f\left(-\frac{1}{2}:-\sqrt{3}\right)=-\frac{5}{4}$.

In conclusion:



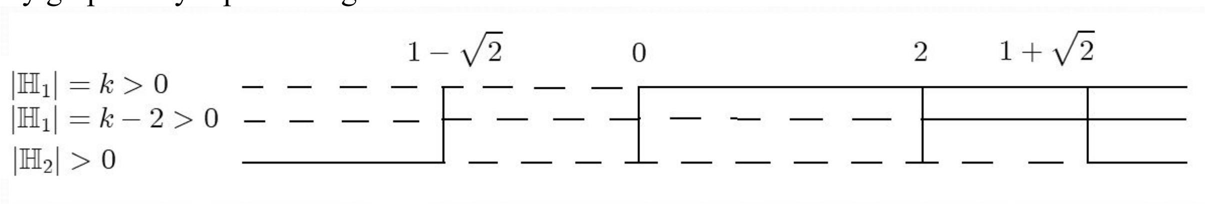
II M 2) Given the function $f(x,y)=x\,y$, let v and w be the unit vectors of $\mathbb{V}=(1,1)$ and $\mathbb{W}=(1,-1)$; determine the point (x_0,y_0) if $\mathcal{D}_v f(x_0,y_0)=\sqrt{2}$ and $\mathcal{D}_w f(x_0,y_0)=0$.

The function $f(x,y)=x\,y$ it is clearly differentiable $\forall\,(x,y)\in\mathbb{R}^2$.
So $D_v f(x_0,y_0)=\nabla f(x_0,y_0)\cdot v$ and $D_w f(x_0,y_0)=\nabla f(x_0,y_0)\cdot w$.
Therefore: $\nabla f(x,y)=(y\,;x)\Rightarrow\nabla f(x_0,y_0)=(y_0,x_0)$; moreover:
$v=\left(\frac{1}{\sqrt{2}};\frac{1}{\sqrt{2}}\right)$ and $w=\left(\frac{1}{\sqrt{2}};-\frac{1}{\sqrt{2}}\right)$, and so we get:

$$\begin{cases} \mathcal{D}_v f(x_0, y_0) = \sqrt{2} \\ \mathcal{D}_w f(x_0, y_0) = 0 \end{cases} \Rightarrow \begin{cases} (y_0, x_0) \cdot \left(\frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}} \right) = \sqrt{2} \\ (y_0, x_0) \cdot \left(\frac{1}{\sqrt{2}}; -\frac{1}{\sqrt{2}} \right) = 0 \end{cases} \Rightarrow \begin{cases} y_0 + x_0 = 2 \\ y_0 - x_0 = 0 \end{cases} \Rightarrow \begin{cases} x_0 = 1 \\ y_0 = 1 \end{cases}.$$

II M 3) In a stationary point of a function of two variables $f(x, y)$ the Hessian matrix is equal to $\mathbb{H} = \begin{Vmatrix} k & 1 \\ 1 & k-2 \end{Vmatrix}$. Determine, on varying the parameter k, the nature of such stationary point.

The first order leading minors are:
$|\mathbb{H}_1| = k > 0$ for $k > 0$ and
$|\mathbb{H}_1| = k - 2 > 0$ for $k > 2$;
The second order leading minor is:
$|\mathbb{H}_2| = k(k-2) - 1 = k^2 - 2k - 1$.
From $k^2 - 2k - 1 = 0$ we get $k = 1 \pm \sqrt{1+1} = 1 \pm \sqrt{2}$.
So $|\mathbb{H}_2| = k^2 - 2k - 1 > 0$ for $k < 1 - \sqrt{2}$ and $k > 1 + \sqrt{2}$.
By graphically representing the situation we have:



and so:

- for $k < 1 - \sqrt{2}$: $\begin{cases} |\mathbb{H}_1| = k < 0 \\ |\mathbb{H}_1| = k - 2 < 0 \\ |\mathbb{H}_2| > 0 \end{cases}$: the stationary point is a maximum point;
- for $k > 1 + \sqrt{2}$: $\begin{cases} |\mathbb{H}_1| = k > 0 \\ |\mathbb{H}_1| = k - 2 > 0 \\ |\mathbb{H}_2| > 0 \end{cases}$: the stationary point is a minimum point;
- for $1 - \sqrt{2} < k < 1 + \sqrt{2}$: $|\mathbb{H}_2| < 0$: the stationary point is a saddle point.

For $k = 1 - \sqrt{2}$; $k = 1 + \sqrt{2}$ it is not possible to establish the nature of the stationary point; for $k = 0$; $k = 2$ $|\mathbb{H}_2| < 0$: the stationary point is a saddle point.

II M 4) Given the system $\begin{cases} f(x, y, z) = \sin(xy) + \cos(xz) = 1 \\ g(x, y, z) = x^3y^2 - xz^3 + zy^3 = 1 \end{cases}$ satisfied at the point $P_0 = (0, 1, 1)$, verify that with it we can define an implicit function $z \to (x, y)$ and then calculate its first derivatives at $z = 1$.

The functions $f(x, y, z)$ and $g(x, y, z)$ are differentiable functions $\forall (x, y, z) \in \mathbb{R}^3$.
It is then:

$$\frac{\partial(f, g)}{\partial(x, y, z)} = \begin{Vmatrix} y\cos(xy) - z\sin(xz) & x\cos(xy) & -x\sin(xz) \\ 3x^2y^2 - z^3 & 2x^3y + 3zy^2 & -3xz^2 + y^3 \end{Vmatrix}$$ and so:

$$\frac{\partial(f, g)}{\partial(x, y, z)}(P_0) = \begin{Vmatrix} 1 & 0 & 0 \\ -1 & 3 & 1 \end{Vmatrix}.$$

Since $\begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} = 3 \neq 0$ we can define an implicit function $z \to (x, y)$ whose derivatives are:

$$\frac{\mathrm{d}x}{\mathrm{d}z} = -\frac{\begin{vmatrix} 0 & 0 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix}} = -\frac{0}{3} = 0\,; \quad \frac{\mathrm{d}y}{\mathrm{d}z} = -\frac{\begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix}} = -\frac{1}{3}\,.$$