QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS
TASK 17/9/2020

I M 1) Calculate the cubic roots of the number $z = \dfrac{1}{1-i} - \dfrac{1}{1+i}$.

Since $z = \dfrac{1}{1-i} - \dfrac{1}{1+i} = \dfrac{1+i-1+i}{(1-i)(1+i)} = \dfrac{2i}{2} = i = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}$ we get:

$\sqrt[3]{i} = \cos\left(\dfrac{\pi}{6} + k\,\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{\pi}{6} + k\,\dfrac{2\pi}{3}\right),\ 0 \le k \le 2\,.$

For $k=0$ we get $z_0 = \cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2} + i\,\dfrac{1}{2}$;

for $k=1$ we get $z_1 = \cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6} = -\dfrac{\sqrt{3}}{2} + i\,\dfrac{1}{2}$;

for $k=2$ we get $z_1 = \cos\dfrac{9\pi}{6} + i\sin\dfrac{9\pi}{6} = \cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2} = -\,i\,.$

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} k & 0 & k \\ 0 & 2 & 0 \\ k & 0 & k \end{Vmatrix}$, determine the values of the parameter k for which the matrix admits a multiple eigenvalue.

The matrix is a symmetric one, therefore certainly it has only real eigenvalues.
From $|\mathbb{A} - \lambda\,\mathbb{I}| = 0$ we get:

$$\begin{vmatrix} k-\lambda & 0 & k \\ 0 & 2-\lambda & 0 \\ k & 0 & k-\lambda \end{vmatrix} = (2-\lambda)\big((k-\lambda)^2 - k^2\big) = (2-\lambda)\big(\lambda^2 - 2k\lambda\big) =$$

$= \lambda\,(2-\lambda)(\lambda-2k) = 0$ and so we get the eigenvalues $\lambda_1 = 0, \lambda_2 = 2$ and $\lambda_3 = 2k$.
So the matrix admits a multiple eigenvalue when $2k = 0 \Rightarrow k = 0 \Rightarrow \lambda_1 = \lambda_2 = 0$ or when $2k = 2 \Rightarrow k = 1 \Rightarrow \lambda_2 = \lambda_3 = 1$.

I M 3) Given the linear system $\begin{cases} x_1 - 2x_3 = 1 \\ 2x_1 - x_2 + k\,x_3 = 2 \\ x_1 - 2x_2 + 6x_3 = k \end{cases}$, check existence and number of its solutions on varying the parameter k.

By Rouchè-Capelli Theorem, from $\mathbb{A} \cdot \mathbb{X} = \mathbb{Y}$ we need to calculate the Rank of the matrix $\mathbb{A}$ and the Rank of the augmented matrix $(\mathbb{A}|\mathbb{Y})$.
By elementary operations on the rows $(R_2 \leftarrow R_2 - 2R_1)$ and $(R_3 \leftarrow R_3 - R_1)$ we get :

$$\begin{Vmatrix} 1 & 0 & -2 & | & 1 \\ 2 & -1 & k & | & 2 \\ 1 & -2 & 6 & | & k \end{Vmatrix} \to \begin{Vmatrix} 1 & 0 & -2 & | & 1 \\ 0 & -1 & k+4 & | & 0 \\ 0 & -2 & 8 & | & k-1 \end{Vmatrix}.$$

By elementary operations on the rows $(R_3 \leftarrow R_3 - 2R_2)$ we get :

$$\begin{Vmatrix} 1 & 0 & -2 & | & 1 \\ 0 & -1 & k+4 & | & 0 \\ 0 & -2 & 8 & | & k-1 \end{Vmatrix} \to \begin{Vmatrix} 1 & 0 & -2 & | & 1 \\ 0 & -1 & k+4 & | & 0 \\ 0 & 0 & -2k & | & k-1 \end{Vmatrix}.$$

And so:

if $-2k = 0 \Rightarrow k = 0 \Rightarrow \text{Rank}(\mathbb{A}) = 2 < \text{Rank}(\mathbb{A}|\mathbb{Y}) = 3$ and so the system has no solutions;

if $-2k \neq 0 \Rightarrow k \neq 0 \Rightarrow \text{Rank}(\mathbb{A}) = 3 = \text{Rank}(\mathbb{A}|\mathbb{Y})$ and so the system has one and only one solution.

I M 4) Determine the matrix $\mathbb{A} = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix}$ knowing that $(1, -1)$ is an eigenvector relative to the eigenvalue $\lambda = 0$ and that $(1,1)$ is an eigenvector relative to $\lambda = 2$.

Since $(1, -1)$ is an eigenvector relative to the eigenvalue $\lambda = 0$ weget:

$$\begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ -1 \end{Vmatrix} = 0 \cdot \begin{Vmatrix} 1 \\ -1 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} a_{11} - a_{12} = 0 \\ a_{21} - a_{22} = 0 \end{cases} \Rightarrow \begin{cases} a_{12} = a_{11} \\ a_{21} = a_{22} \end{cases};$$

Since $(1,1)$ is an eigenvector relative to the eigenvalue $\lambda = 2$ weget:

$$\begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} = 2 \cdot \begin{Vmatrix} 1 \\ 1 \end{Vmatrix} = \begin{Vmatrix} 2 \\ 2 \end{Vmatrix} \Rightarrow \begin{cases} a_{11} + a_{12} = 2 \\ a_{21} + a_{22} = 2 \end{cases} \Rightarrow \begin{cases} a_{12} = 2 - a_{11} \\ a_{21} = 2 - a_{22} \end{cases};$$ from these

we get: $\begin{cases} a_{11} = 2 - a_{11} \\ a_{22} = 2 - a_{22} \end{cases} \Rightarrow \begin{cases} a_{11} = 1 \\ a_{22} = 1 \end{cases} \Rightarrow \begin{cases} a_{12} = 1 \\ a_{21} = 1 \end{cases} \Rightarrow \mathbb{A} = \begin{Vmatrix} 1 & 1 \\ 1 & 1 \end{Vmatrix}$.

II M 1) Solve the problem : $\begin{cases} \text{Max/min } f(x,y) = x^2 + y^2 - y \\ \text{u.c.:} \begin{cases} y \leq 1 - 2x \\ x \geq 0 \\ y \geq 0 \end{cases} \end{cases}$.

The objective function of the problem is a continuous function, the constraints are linear functions and they define a feasible region which is a compact and bounded set as it is a triangle, and so we can apply Weierstrass Theorem. To solve the problem we don't use the Kuhn-Tucker Theorem.

We determine the stationary points of the function.

Applying the first order conditions we have:

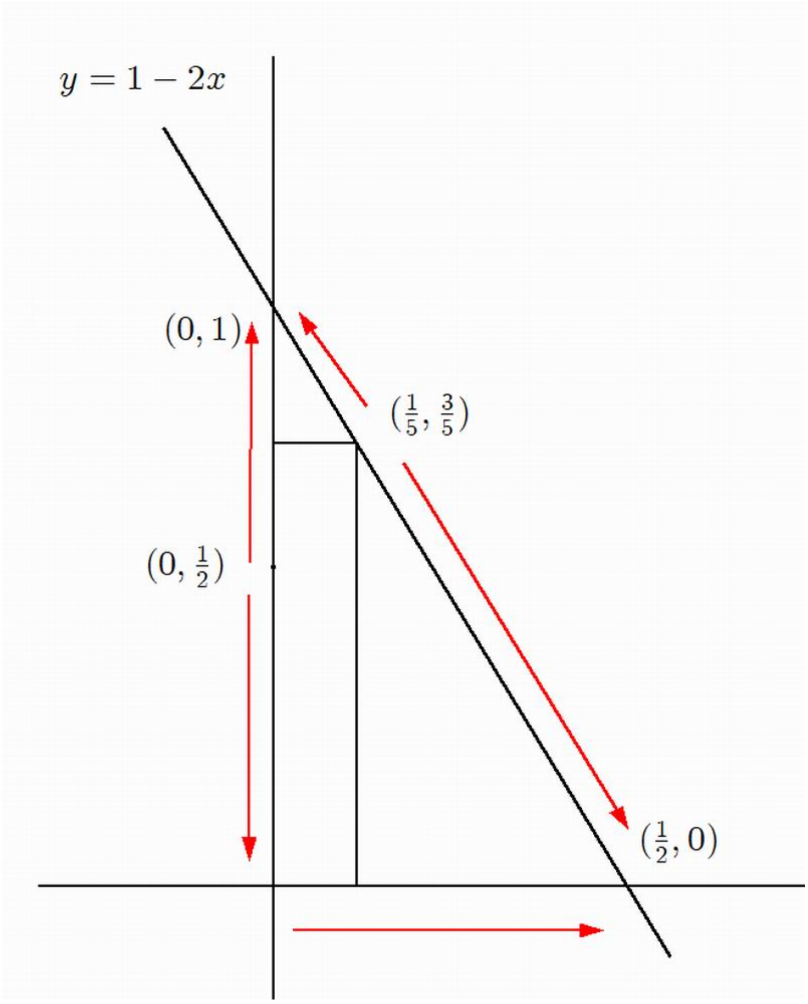
$\nabla f(x,y) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 2x = 0 \\ f'_y = 2y - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = \frac{1}{2} \end{cases}$. Since $\mathbb{H}(x,y) = \begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix} = \mathbb{H}\left(0, \frac{1}{2}\right)$, immediatly we see that $\left(0, \frac{1}{2}\right)$ is a minimum point.

For $x = 0$ we get $f(0,y) = y^2 - y \Rightarrow f'(0,y) = 2y - 1 \geq 0$ for $y \geq \frac{1}{2}$ and so the result that we have already found: $\left(0, \frac{1}{2}\right)$ is a minimum point.

For $y = 0$ we get $f(x,0) = x^2 \Rightarrow f'(x,0) = 2x \geq 0$ for $x \geq 0$ and so the function is always increasing from $x = 0$ to $x = \frac{1}{2}$.

For $y = 1 - 2x$ we get $f(x, 1-2x) = 5x^2 - 2x \Rightarrow f'(x, 1-2x) = 10x - 2 \geq 0$ for $x \geq \frac{1}{5}$ and so the function is decreasing from $x = 0$ to $x = \frac{1}{5}$ and then is increasing from $x = \frac{1}{5}$ to $x = \frac{1}{2}$, so $x = \frac{1}{5}$ is a minimum point and so $\left(\frac{1}{5}, \frac{3}{5}\right)$ is a minimum point.

By representing with a figure the results found we have:



So $(0,1)$ is a maximum point, with $f(0,1) = 0$; relative maximum point;

$\left(\frac{1}{2}, 0\right)$ is a maximum point, with $f\left(\frac{1}{2}, 0\right) = \frac{1}{4}$; absolute maximum point;

$\left(0, \frac{1}{2}\right)$ is a minimum point, with $f\left(0, \frac{1}{2}\right) = -\frac{1}{4}$; absolute minimum point;

$\left(\frac{1}{5}, \frac{3}{5}\right)$ is a minimum point, with $f\left(\frac{1}{5}, \frac{3}{5}\right) = -\frac{1}{5}$; relative minimum point.

II M 2) Given the function $f(x,y) = x^2 - xy^2$, and the unit vector $v = (\cos\alpha, \sin\alpha)$, calculate $\mathcal{D}_v f(1,1)$ and $\mathcal{D}^2_{v,v} f(1,1)$. And then calculate $\mathcal{D}^2_{v,v} f(1,1)$ when $\alpha = \frac{\pi}{4}$.

The function $f(x,y) = x^2 - xy^2$ it is clearly differentiable of every order $\forall (x,y) \in \mathbb{R}^2$.
So $D_v f(1,1) = \nabla f(1,1) \cdot v$ and $\mathcal{D}^2_{v,v} f(1,1) = v \cdot \mathbb{H} f(1,1) \cdot v^T$.
From $\nabla f(x,y) = \left(2x - y^2, \, -2xy\right)$ we get $\nabla f(1,1) = (1, -2)$ and so:
$\mathcal{D}_v f(1,1) = (1, -2) \cdot (\cos\alpha, \sin\alpha) = \cos\alpha - 2\sin\alpha$. Then we get :

$$\mathbb{H}(x,y) = \begin{Vmatrix} 2 & -2y \\ -2y & -2x \end{Vmatrix} \Rightarrow \mathbb{H}(1,1) = \begin{Vmatrix} 2 & -2 \\ -2 & -2 \end{Vmatrix} \text{ and so:}$$

$$\mathcal{D}^2_{v,v} f(1,1) = \begin{Vmatrix} \cos\alpha & \sin\alpha \end{Vmatrix} \cdot \begin{Vmatrix} 2 & -2 \\ -2 & -2 \end{Vmatrix} \cdot \begin{Vmatrix} \cos\alpha \\ \sin\alpha \end{Vmatrix} \Rightarrow$$

$$\mathcal{D}^2_{v,v} f(1,1) = 2\cos^2\alpha - 2\sin^2\alpha - 4\cos\alpha\sin\alpha = 2\cos 2\alpha - 2\sin 2\alpha .$$

Finally, for $\alpha = \dfrac{\pi}{4}$ we get: $\mathcal{D}^2_{v,v} f(1,1) = 2\cos\dfrac{\pi}{2} - 2\sin\dfrac{\pi}{2} = -2$.

II M 3) Given the function $f(x,y) = x^2 + x^2 y + y^2$ determine existence and nature of its stationary points.

Applying the first order conditions we have:

$\nabla f(x,y) = \mathbb{O} \Rightarrow \begin{cases} f'_x = 2x + 2xy = 2x(1+y) = 0 \\ f'_y = x^2 + 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \cup \begin{cases} x^2 = 2 \\ y = -1 \end{cases}$ and so the solutions $\begin{cases} x = 0 \\ y = 0 \end{cases}$, $\begin{cases} x = \sqrt{2} \\ y = -1 \end{cases}$ and $\begin{cases} x = -\sqrt{2} \\ y = -1 \end{cases}$.

Since $\mathbb{H}(x,y) = \begin{Vmatrix} 2+2y & 2x \\ 2x & 2 \end{Vmatrix}$ we get:

- $\mathbb{H}(0,0) = \begin{Vmatrix} 2 & 0 \\ 0 & 2 \end{Vmatrix} \Rightarrow \begin{cases} |\mathbb{H}_1| = 2 > 0 \\ |\mathbb{H}_2| = 4 > 0 \end{cases}$ and so $(0,0)$ is a minimum point;
- $\mathbb{H}\left(\sqrt{2}, -1\right) = \begin{Vmatrix} 0 & \sqrt{2} \\ \sqrt{2} & 2 \end{Vmatrix} \Rightarrow |\mathbb{H}_2| = -2 < 0$ so $\left(\sqrt{2}, -1\right)$ is a saddle point;
- $\mathbb{H}\left(-\sqrt{2}, -1\right) = \begin{Vmatrix} 0 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{Vmatrix} \Rightarrow |\mathbb{H}_2| = -2 < 0$ so $\left(-\sqrt{2}, -1\right)$ is a saddle point.

II M 4) Given the equation $f(x,y) = x\,e^{x+y} - y\,e^{x-y} = 0$, verify if at $\mathrm{P}_0 = (0,0)$ it is possible to define an implicit function having y as dependent variable of x and then calculate the partial derivative of the first order of $y(x)$ at $x = 0$.

The functions$f(x,y)$ is a differentiable function $\forall\,(x,y) \in \mathbb{R}^2$, $f(0,0) = 0 - 0 = 0$.
It is then: $\nabla f(x,y) = (e^{x+y} + x\,e^{x+y} - y\,e^{x-y}; x\,e^{x+y} - e^{x-y} + y\,e^{x-y})$ from which:
$\nabla f(x,y) = ((1+x)\,e^{x+y} - y\,e^{x-y}; x\,e^{x+y} - (1-y)\,e^{x-y})$.
So $\nabla f(0,0) = (1-0; 0-1) = (1, -1)$. Since $f'_y(0,0) = -1 \neq 0$ we can define an implicit function $x \to y(x)$ whose first order derivative is:

$$\frac{\mathrm{d}y}{\mathrm{d}x}(0) = -\frac{f'_x(0,0)}{f'_y(0,0)} = -\frac{1}{-1} = 1.$$