QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS
TASK 6/10/2020

I M 1) Calcolate $\sqrt{\left(1+\sqrt{3}\,i\right)^3}$.

Since $1+\sqrt{3}\,i = 2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right) = 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$ we get:

$\left(1+\sqrt{3}\,i\right)^3 = 8\left(\cos 3\,\frac{\pi}{3}+i\sin 3\,\frac{\pi}{3}\right) = 8\left(\cos\pi+i\sin\pi\right) = -8$.

And finally:

$\sqrt{\left(1+\sqrt{3}\,i\right)^3} = \sqrt{-8} = \sqrt{8}\left[\cos\left(\frac{\pi}{2}+k\,\frac{2\pi}{2}\right)+i\sin\left(\frac{\pi}{2}+k\,\frac{2\pi}{2}\right)\right],\ 0\le k\le 1$.

For $k=0$ we get $z_0 = \sqrt{8}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) = 2\sqrt{2}\,i$;

for $k=1$ we get $z_1 = \sqrt{8}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right) = -2\sqrt{2}\,i$.

I M 2) Given the linear map $f:\mathbb{R}^4\to\mathbb{R}^3$ having matrix $\mathbb{A} = \begin{Vmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 2 \\ 2 & 0 & k & m \end{Vmatrix}$, determine the value of the parameters k and m if the Kernel and the Image have the same dimension. For such values find a basis for the Kernel of the linear map generated by $\mathbb{A}$.

To get $\text{Dim(Imm)} = \text{Dim(Ker)}$ we need $\text{Rank}(\mathbb{A}) = 2$.

By elementary operations on the rows $(R_3 \leftarrow R_3 - 2R_1)$ we get :

$$\begin{Vmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 2 \\ 2 & 0 & k & m \end{Vmatrix} \to \begin{Vmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & k-4 & m-2 \end{Vmatrix};$$

By elementary operations on the rows $(R_3 \leftarrow R_3 - R_2)$ we get :

$$\begin{Vmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & k-4 & m-2 \end{Vmatrix} \to \begin{Vmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & k-5 & m-4 \end{Vmatrix}.$$

So $\text{Dim(Imm)} = \text{Rank}(\mathbb{A}) = \text{Dim(Ker)} = 2$ for $k=5$ and $m=4$.

And so $\mathbb{A} = \begin{Vmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 2 \\ 2 & 0 & 5 & 4 \end{Vmatrix}$.

To find a basis for the Kernel of the linear map generated by $\mathbb{A}$ we must firstly solve the system: $\mathbb{A}\cdot\mathbb{X} = \mathbb{O} \Rightarrow \begin{Vmatrix} 1 & -1 & 2 & 1 \\ 0 & 2 & 1 & 2 \\ 2 & 0 & 5 & 4 \end{Vmatrix} \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix}$.

So we get the system: $\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 0 \\ 2x_2 + x_3 + 2x_4 = 0 \\ 2x_1 + 5x_3 + 4x_4 = 0 \end{cases}$ which however, for the elementary operations on the rows that we have already done, is equivalent to the system:

$$\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 0 \\ 2x_2 + x_3 + 2x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_4 = -x_1 + x_2 - 2x_3 \\ -2x_1 + 4x_2 - 3x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_3 = -\frac{2}{3}x_1 + \frac{4}{3}x_2 \\ x_4 = \frac{1}{3}x_1 - \frac{5}{3}x_2 \end{cases}.$$

Each vector of the Kernel can be represented as $\mathbb{X} = \left(x_1, x_2, -\frac{2}{3}x_1 + \frac{4}{3}x_2, \frac{1}{3}x_1 - \frac{5}{3}x_2\right)$.

If we choose $x_1 = 3$ and $x_2 = 0$ we get the vector $\mathbb{X}_1 = (3, 0, -2, 1)$.

If we choose $x_1 = 0$ and $x_2 = 3$ we get the vector $\mathbb{X}_2 = (0, 3, 4, -5)$.

So a basis for the Kernel is: $\mathbb{X} = \{(3, 0, -2, 1), (0, 3, 4, -5)\}$.

I M 3) Given the matrix $\mathbb{A} = \begin{Vmatrix} 2 & 2 & -1 \\ 3 & 0 & -2 \\ k & 2 & 1 \end{Vmatrix}$, find the value of the parameter k knowing that the matrix admits the eigenvalue $\lambda = -1$ and then determine, for this value of the parameter, if the matrix is diagonalizable or not.

From $|\mathbb{A} - \lambda\mathbb{I}| = 0$ we get:

$$\begin{vmatrix} 2-\lambda & 2 & -1 \\ 3 & -\lambda & -2 \\ k & 2 & 1-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda-k & 0 & \lambda-2 \\ 3 & -\lambda & -2 \\ k & 2 & 1-\lambda \end{vmatrix} =$$

$$= (2-\lambda-k)\left(\lambda^2 - \lambda + 4\right) + (\lambda - 2)(6 + k\lambda) = 0.$$

If $\lambda = -1$ we get $(3-k)(6) + (-3)(6-k) = 18 - 6k - 18 + 3k = 0 \Rightarrow k = 0$.

So $\mathbb{A} = \begin{Vmatrix} 2 & 2 & -1 \\ 3 & 0 & -2 \\ 0 & 2 & 1 \end{Vmatrix}$. And also:

$$|\mathbb{A} - \lambda\mathbb{I}| = (2-\lambda)\left(\lambda^2 - \lambda + 4\right) + (\lambda - 2)(6) = (2-\lambda)\left(\lambda^2 - \lambda - 2\right) = 0 \Rightarrow$$

$$\Rightarrow (2-\lambda)(\lambda-2)(\lambda+1) = 0 \Rightarrow \lambda_1 = \lambda_2 = 2, \lambda_3 = -1.$$

To check if the matrix is diagonalizable or not we have to find the Rank of $\|\mathbb{A} - 2\,\mathbb{I}\|$:

$\|\mathbb{A} - 2\,\mathbb{I}\| = \begin{Vmatrix} 0 & 2 & -1 \\ 3 & -2 & -2 \\ 0 & 2 & -1 \end{Vmatrix}$. Since $\begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} = -6 \neq 0$ it is $\text{Rank}(\mathbb{A} - 2\,\mathbb{I}) = 2$ for which it is $m_2^g = 3 - 2 = 1 < m_2^a = 2$ and so the matrix is not a diagonalizable one.

I M 4) Find the coordinates of the vector $\mathbb{X}$ in the basis $\mathbb{V} = \{(1,0,0), (1,1,0), (1,0,1)\}$ if it has coordinates $(2, 0, -1)$ in the basis $\mathbb{W} = \{(1,1,0), (1,-1,1), (1,0,1)\}$.

Since $\mathbb{X}$ has coordinates $(2, 0, -1)$ in the basis $\mathbb{W} = \{(1,1,0), (1,-1,1), (1,0,1)\}$, it is:

$$\mathbb{X} = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} 2 \\ 0 \\ -1 \end{Vmatrix} = \begin{Vmatrix} 1 \\ 2 \\ -1 \end{Vmatrix}.$$

To find the coordinates of the vector $\mathbb{X}$ in the basis $\mathbb{V} = \{(1,0,0), (1,1,0), (1,0,1)\}$ we have to solve the system:

$$\begin{Vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 1 \\ 2 \\ -1 \end{Vmatrix} \Rightarrow \begin{cases} x + y + z = 1 \\ y = 2 \\ z = -1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 2 \\ z = -1 \end{cases}.$$

II M 1) Solve the problem : $\begin{cases} \text{Max/min } f(x,y) = x^2 - y^2 \\ \text{u.c. : } x^2 \le y \le 1 \end{cases}$.

The objective function of the problem is a continuous function, the constraints define a feasible region which is a compact and bounded set, and so we can apply Weierstrass Theorem. Surely the function admits maximum value and minimum value.
To solve the problem we use the Kuhn-Tucker conditions.

We write the problem as $\begin{cases} \text{Max/min } f(x,y) = x^2 - y^2 \\ \text{u.c.: } \begin{cases} x^2 - y \le 0 \\ y - 1 \le 0 \end{cases} \end{cases}$

$\Lambda(x, y, \lambda_1, \lambda_2) = x^2 - y^2 - \lambda_1 (x^2 - y) - \lambda_2 (y - 1)$.
By applying the first order conditions we have:
1) case $\lambda_1 = 0, \lambda_2 = 0$:

$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = -2y = 0 \\ x^2 - y \le 0 \\ y - 1 \le 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ 0 \le 0 \\ -1 \le 0 \end{cases}$. Since $\mathbb{H}(x,y) = \begin{Vmatrix} 2 & 0 \\ 0 & -2 \end{Vmatrix} = \mathbb{H}(0,0)$ we have that, globally, the point $(0,0)$ is a saddle point. We will study it more precisely later.

2) case $\lambda_1 \ne 0, \lambda_2 = 0$:

$\begin{cases} \Lambda'_x = 2x - 2\lambda_1 x = 2x(1 - \lambda_1) = 0 \\ \Lambda'_y = -2y + \lambda_1 = 0 \\ y = x^2 \\ y \le 1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ \lambda_1 = 0 \\ 0 \le 1 \end{cases}$: we will study it more precisely later;

$\begin{cases} \Lambda'_x = 2x - 2\lambda_1 x = 2x(1 - \lambda_1) = 0 \\ \Lambda'_y = -2y + \lambda_1 = 0 \\ y = x^2 \\ y \le 1 \end{cases} \Rightarrow \begin{cases} x^2 = \frac{1}{2} \\ y = \frac{1}{2} \\ \lambda_1 = 1 \\ \frac{1}{2} \le 1 \end{cases} \Rightarrow \begin{cases} x = \frac{1}{\sqrt{2}} \\ y = \frac{1}{2} \\ \lambda_1 = 1 \\ \frac{1}{2} \le 1 \end{cases} \cup \begin{cases} x = -\frac{1}{\sqrt{2}} \\ y = \frac{1}{2} \\ \lambda_1 = 1 \\ \frac{1}{2} \le 1 \end{cases}$.

Since $\lambda_1 > 0$ the points $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ may be maximum points.

3) case $\lambda_1 = 0, \lambda_2 \ne 0$:

$\begin{cases} \Lambda'_x = 2x = 0 \\ \Lambda'_y = -2y - \lambda_2 = 0 \\ y = 1 \\ x^2 \le y \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_2 = -2 < 0 \\ 0 \le 1 \end{cases}$.

Since $\lambda_2 < 0$ the point $(0,1)$ may be a minimum point.

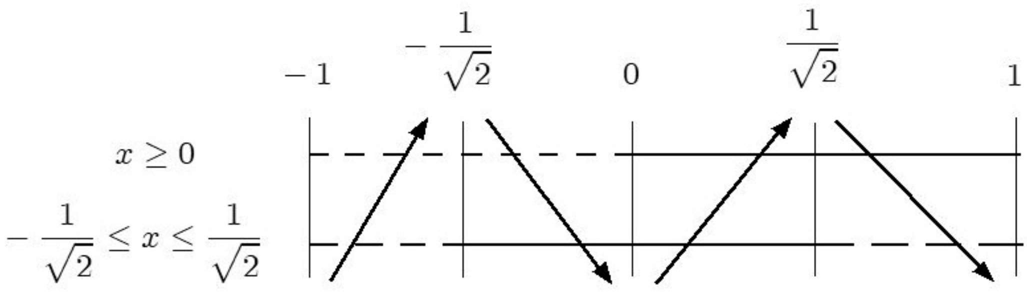
4) case $\lambda_1 \ne 0, \lambda_2 \ne 0$:

$\begin{cases} \Lambda'_x = 2x - 2\lambda_1 x = 0 \\ \Lambda'_y = -2y + \lambda_1 - \lambda_2 = 0 \\ y = x^2 \\ y = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \\ 2 - 2\lambda_1 = 0 \\ \lambda_1 - \lambda_2 = 2 \end{cases} \cup \begin{cases} x = -1 \\ y = 1 \\ -2 + 2\lambda_1 = 0 \\ \lambda_1 - \lambda_2 = 2 \end{cases} \Rightarrow$

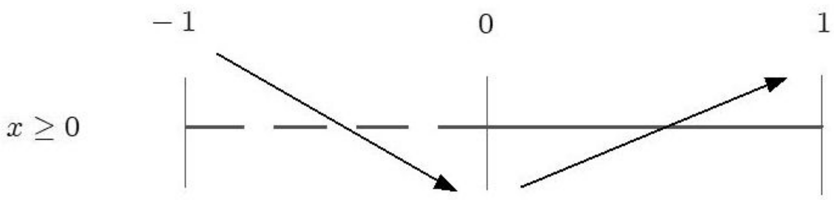
$$\Rightarrow \begin{cases} x = 1 \\ y = 1 \\ \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases} \cup \begin{cases} x = -1 \\ y = 1 \\ \lambda_1 = 1 \\ \lambda_2 = -1 \end{cases}$$

. The points $(1,1)$ and $(-1,1)$ are neither maximum nor minimum points.

Let's study the objective function on the constraint $y = x^2$.
It is $f\left(x, x^2\right) = x^2 - x^4 \Rightarrow f'(x) = 2x - 4x^3 = 2x\left(1 - 2x^2\right) \geq 0$.
It is $1 - 2x^2 \geq 0 \Rightarrow x^2 \leq \frac{1}{2} \Rightarrow -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$ and so:

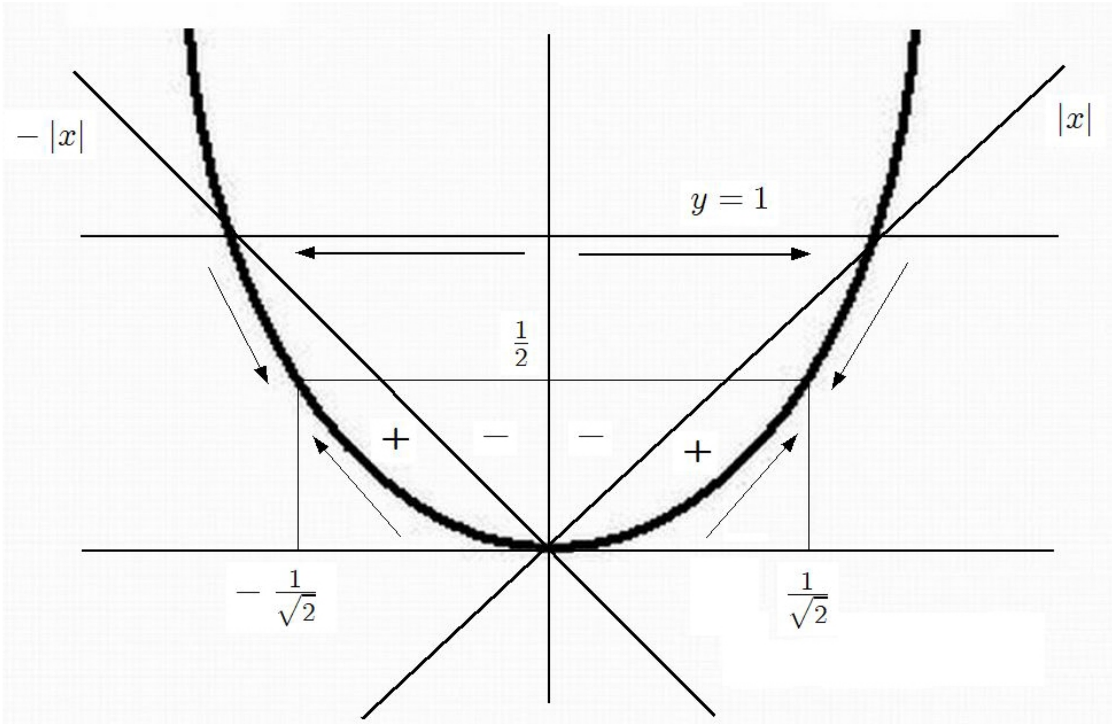


The points $\left(\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ are maximum points with $f\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \frac{1}{4}$.
Let's study the objective function on the constraint $y = 1$.
It is $f(x, 1) = x^2 - 1 \Rightarrow f'(x) = 2x \geq 0$ for $x \geq 0$.



The point $(0,1)$ is a minimum point with $f(0,1) = -1$.

Finally we study the point $(0,0)$. It is $f(0,0) = 0$. Then $f(x,y) = x^2 - y^2 \geq 0$ for $y^2 \leq x^2$ satisfied for $-|x| \leq y \leq |x|$. In each neighborhood of the point $(0,0)$ we have both positive and negative values and therefore $(0,0)$ is a saddle point.



II M 2) Given the function $f(x,y,z)=x^2-xy+y^2+z^2$ determine the nature of its stationary point.

We determine the stationary points of the function. Applying the first order conditions we have:

$$\nabla f(x,y,z)=\mathbb{O}\Rightarrow\begin{cases} f'_x=2x-y=0 \\ f'_y=\ -x+2y=0 \\ f'_z=2z=0 \end{cases}\Rightarrow\begin{cases} x=0 \\ y=0 \\ z=0 \end{cases}.$$

Then we apply the second order conditions: $\mathbb{H}(x,y,z)=\left\|\begin{matrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{matrix}\right\|=\mathbb{H}(0,0,0)$.

Since: $\begin{cases} |\mathbb{H}_1|=2>0 \\ |\mathbb{H}_2|=3>0 \\ |\mathbb{H}_3|=6>0 \end{cases}$ the point $(0,0,0)$ is a minimum point.

II M 3) Given the function $f(x,y)=e^{x^2+y^2}$ and the unit vector $v=(\cos\alpha,\sin\alpha)$ calculate its directional derivatives $\mathcal{D}^2_{vv}f(0,0)$.

The function $f(x,y)=e^{x^2+y^2}$ it is clearly differentiable of every order $\forall\,(x,y)\in\mathbb{R}^2$.
So $\mathcal{D}^2_{v,v}f(0,0)=v\cdot\mathbb{H}f(0,0)\cdot v^T$.
From $\nabla f(x,y)=\left(2x\,e^{x^2+y^2};2y\,e^{x^2+y^2}\right)$ we get :

$$\mathbb{H}(x,y)=\left\|\begin{matrix} 2(1+2x^2)\,e^{x^2+y^2} & 4xy\,e^{x^2+y^2} \\ 4xy\,e^{x^2+y^2} & 2(1+2y^2)\,e^{x^2+y^2} \end{matrix}\right\|\Rightarrow\mathbb{H}(0,0)=\left\|\begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix}\right\|$$ and so:

$$\mathcal{D}^2_{v,v}f(0,0)=\|\cos\alpha\quad\sin\alpha\,\|\cdot\left\|\begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix}\right\|\cdot\left\|\begin{matrix} \cos\alpha \\ \sin\alpha \end{matrix}\right\|=2\cos^2\alpha+2\sin^2\alpha=2\,.$$

II M 4) Given the equation $f(x,y) = x\sin y - y\cos x = 0$, verify if at $P_0 = (0,0)$ it is possible to define an implicit function having y as dependent variable of x and then calculate the partial derivative of the first order of $y(x)$ at $x = 0$.

The function $f(x,y)$ is a differentiable function $\forall\,(x,y) \in \mathbb{R}^2$, $f(0,0) = 0 - 0 = 0$.
It is then: $\nabla f(x,y) = (\sin y + y\sin x; x\cos y - \cos x)$.
So $\nabla f(0,0) = (0; -1)$. Since $f'_y(0,0) = -1 \neq 0$ we can define an implicit function $x \to y(x)$ whose first order derivative is:

$$\frac{\mathrm{d}y}{\mathrm{d}x}(0) = -\frac{f'_x(0,0)}{f'_y(0,0)} = -\frac{0}{-1} = 0.$$