

QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS
TASK 13/1/2021

I M 1) If $e^z = \frac{\sqrt[3]{e}}{2}(\sqrt{3} - i)$, find z .

Applying the definition of complex exponential: $e^{x+iy} = e^x (\cos y + i \sin y)$ and since :

$$e^z = \frac{\sqrt[3]{e}}{2}(\sqrt{3} - i) = e^{\frac{1}{3}} \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i \right) = e^{\frac{1}{3}} \left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} \right) \text{ we get :}$$

$$e^z = e^{\frac{1}{3} + i \frac{11\pi}{6}} \text{ and so } z = \frac{1}{3} + i \frac{11\pi}{6}.$$

I M 2) Given the vectors $\mathbb{X}_1 = (1, 0, -2, 2)$, $\mathbb{X}_2 = (2, 1, -1, 2)$ and $\mathbb{X}_3 = (1, 2, 4, k)$, check, depending on the parameter k , when they are linearly independent.

We solve the problem by calculating the rank of the matrix $\mathbb{M} = \begin{vmatrix} 1 & 0 & -2 & 2 \\ 2 & 1 & -1 & 2 \\ 1 & 2 & 4 & k \end{vmatrix}$.

If the vectors are linearly independent, the rank must be equal to 3; if, on the other hand, the vectors are linearly dependent, the rank must be less than 3.

Since $\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1 \neq 0$, the rank is at least equal to 2.

By elementary operations on the rows $(R_2 \leftarrow R_2 - 2R_1)$, $(R_3 \leftarrow R_3 - R_1)$, we get :

$$\begin{vmatrix} 1 & 0 & -2 & 2 \\ 2 & 1 & -1 & 2 \\ 1 & 2 & 4 & k \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & k-2 \end{vmatrix}.$$

By elementary operations on the rows $(R_3 \leftarrow R_3 - 2R_2)$, we get :

$$\begin{vmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & k-2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & k+2 \end{vmatrix}.$$

Therefore, if $k \neq -2$ the rank is equal to 3 and the vectors are linearly independent, if instead $k = -2$ the rank is equal to 2 and the vectors are dependent.

I M 3) Determine the only value of the parameter k for which the matrix $\begin{vmatrix} 1 & 2 & 3 \\ 0 & k & 2 \\ 0 & 0 & 1 \end{vmatrix}$ is diagonalizable.

From $|\mathbb{A} - \lambda \mathbb{I}| = 0$ we get:

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & k-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda)(k-\lambda)(1-\lambda) = 0. \text{ So } \lambda_1 = \lambda_2 = 1, \lambda_3 = k.$$

If $k = 1$ we get $m_1^a = 3$ and $\|\mathbb{A} - 1 \cdot \mathbb{I}\| = \begin{vmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{vmatrix}$ and since $\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = 4 \neq 0$, it is

$\text{Rank}(\mathbb{A} - 1 \cdot \mathbb{I}) = 2$, so $m_1^g = 3 - 2 = 1 < m_1^a = 3$ and the matrix is not diagonalizable.

If $k \neq 1$ it is $m_1^a = 2$ and $\|\mathbb{A} - 1 \cdot \mathbb{I}\| = \begin{vmatrix} 0 & 2 & 3 \\ 0 & k-1 & 2 \\ 0 & 0 & 0 \end{vmatrix}$; if $\begin{vmatrix} 2 & 3 \\ k-1 & 2 \end{vmatrix} = 7 - 3k = 0$

then $\text{Rank}(\mathbb{A} - 1 \cdot \mathbb{I}) = 1$, so $m_1^g = 3 - 1 = 2 = m_1^a$ and the matrix is diagonalizable.

If $k \neq \frac{7}{3}$ then $\begin{vmatrix} 2 & 3 \\ k-1 & 2 \end{vmatrix} \neq 0$, $\text{Rank}(\mathbb{A} - 1 \cdot \mathbb{I}) = 2$, so $m_1^g = 3 - 2 = 1 < m_1^a$ and the matrix is not diagonalizable. Therefore the only value of the parameter k for which the matrix is diagonalizable is $k = \frac{7}{3}$.

I M 4) Consider the linear map $f : \mathbb{R}^4 \rightarrow \mathbb{R}^3$, $\mathbb{Y} = \mathbb{A} \cdot \mathbb{X}$ whith $\mathbb{A} = \begin{vmatrix} 1 & -1 & m & 1 \\ 0 & 2 & 1 & k \\ 1 & 1 & 2k & m \end{vmatrix}$.

Since $f(1, 1, 1, 1) = (4, 5, 9)$, determine the values of the parameters m and k and then find the dimensions of the Kernel and of the Image of this linear map.

From $f(1, 1, 1, 1) = (4, 5, 9)$ we get:

$$\begin{vmatrix} 1 & -1 & m & 1 \\ 0 & 2 & 1 & k \\ 1 & 1 & 2k & m \end{vmatrix} \cdot \begin{vmatrix} 1 \\ 1 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 4 \\ 5 \\ 9 \end{vmatrix} \Rightarrow \begin{cases} 1 - 1 + m + 1 = 4 \\ 0 + 2 + 1 + k = 5 \\ 1 + 1 + 2k + m = 9 \end{cases} \Rightarrow \begin{cases} m = 3 \\ k = 2 \\ 2 + 4 + 3 = 9 \end{cases} .$$

So $\mathbb{A} = \begin{vmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & 4 & 3 \end{vmatrix}$. By elementary operations on the rows $(R_3 \leftarrow R_3 - R_1)$ and

then $(R_3 \leftarrow R_3 - R_2)$, we get :

$$\begin{vmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & 4 & 3 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 2 & 1 & 2 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & -1 & 3 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} . \text{ Since } \begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix} \neq 0$$

we get $\text{Rank}(\mathbb{A}) = 2$ and so $\text{Dim(Imm)} = 2$ and $\text{Dim(Ker)} = 4 - 2 = 2$.

II M 1) Given $f(x, y) = x^2 - 3xy + 2y^2$, v and w the unit vectors of $(1, 1)$ and $(1, -1)$, since $\mathcal{D}_v f(P_0) = \sqrt{2}$ and $\mathcal{D}_w f(P_0) = 2\sqrt{2}$, determine P_0 and then calcolate $D_{v,w}^2 f(P_0)$.

The function $f(x, y) = x^2 - 3xy + 2y^2$ ist twice differentiable $\forall (x, y) \in \mathbb{R}^2$.

So $\mathcal{D}_v f(P_0) = \nabla f(P_0) \cdot v$ and $\mathcal{D}_{v,w}^2 f(P_0) = v \cdot \mathbb{H}f(P_0) \cdot w^T$.

From $\nabla f(x, y) = (2x - 3y; 4y - 3x)$ we get :

$$\begin{cases} \mathcal{D}_v f(P_0) = \nabla f(P_0) \cdot v = (2x - 3y; 4y - 3x) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = \sqrt{2} \\ \mathcal{D}_w f(P_0) = \nabla f(P_0) \cdot w = (2x - 3y; 4y - 3x) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = 2\sqrt{2} \\ \begin{cases} 2x - 3y + 4y - 3x = 2 \\ 2x - 3y - 4y + 3x = 4 \end{cases} \Rightarrow \begin{cases} y - x = 2 \\ 5x - 7y = 4 \end{cases} \Rightarrow \begin{cases} y = x + 2 \\ 5x - 7x - 14 = 4 \end{cases} \Rightarrow \begin{cases} x = -9 \\ y = -7 \end{cases} \end{cases} \Rightarrow$$

Therefore $P_0 = (-9, -7)$. It is then $\mathbb{H}(x, y) = \begin{vmatrix} 2 & -3 \\ -3 & 4 \end{vmatrix} = \mathbb{H}(-9, -7)$ and so:

$$\mathcal{D}_{v,w}^2 f(P_0) = \left\| \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right\| \cdot \left\| \begin{matrix} 2 & -3 \\ -3 & 4 \end{matrix} \right\| \cdot \left\| \begin{matrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{matrix} \right\| = \left\| \frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \right\| \cdot \left\| \begin{matrix} \frac{5}{\sqrt{2}} \\ -\frac{7}{\sqrt{2}} \end{matrix} \right\| =$$

$$= \frac{5}{2} - \frac{7}{2} = -1.$$

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x, y) = x(y+1) \\ \text{u.c.: } \begin{cases} x \leq 1 - y^2 \\ 1 \leq x + y \end{cases} \end{cases}$.

The objective function of the problem is a continuous function, the constraints define a feasible region which is a compact and bounded set, and so we can apply Weierstrass Theorem. Surely the function admits maximum value and minimum value.

To solve the problem we use the Kuhn-Tucker conditions.

We write the problem as $\begin{cases} \text{Max/min } f(x, y) = x(y+1) \\ \text{u.c.: } \begin{cases} x + y^2 - 1 \leq 0 \\ 1 - x - y \leq 0 \end{cases} \end{cases}$.

We form the Lagrangian function:

$$\Lambda(x, y, \lambda_1, \lambda_2) = x(y+1) - \lambda_1(x + y^2 - 1) - \lambda_2(1 - x - y).$$

By applying the first order conditions we have:

1) case $\lambda_1 = 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = y + 1 = 0 \\ \Lambda'_y = x = 0 \\ x + y^2 - 1 \leq 0 \\ 1 - x - y \leq 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \\ 0 + 1 - 1 \leq 0 \\ 1 - 0 + 1 \leq 0 : \text{NO} \end{cases} \Rightarrow (0, -1) \notin \mathcal{E}.$$

2) case $\lambda_1 \neq 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = y + 1 - \lambda_1 = 0 \\ \Lambda'_y = x - 2\lambda_1 y = 0 \\ x = 1 - y^2 \\ 1 - x - y \leq 0 \end{cases} \Rightarrow \begin{cases} y = \lambda_1 - 1 \\ x = 2\lambda_1(\lambda_1 - 1) = 2\lambda_1^2 - 2\lambda_1 \\ 2\lambda_1^2 - 2\lambda_1 = 1 - (\lambda_1 - 1)^2 \\ 1 - x - y \leq 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} y = \lambda_1 - 1 \\ x = 2\lambda_1(\lambda_1 - 1) = 2\lambda_1^2 - 2\lambda_1 \\ 3\lambda_1^2 - 4\lambda_1 = \lambda_1(3\lambda_1 - 4) = 0 \\ 1 - x - y \leq 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \\ \lambda_1 = 0 \\ 1 - 0 + 1 \leq 0; \text{NO} \end{cases} \cup \begin{cases} x = \frac{8}{9} \\ y = \frac{1}{3} \\ \lambda_1 = \frac{4}{3} > 0 \\ 1 - \frac{8}{9} - \frac{1}{3} \leq 0 \end{cases};$$

Since $\lambda_1 = \frac{4}{3} > 0$ the point $\left(\frac{8}{9}, \frac{1}{3}\right)$ could be a Maximum point.

3) case $\lambda_1 = 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = y + 1 + \lambda_2 = 0 \\ \Lambda'_y = x + \lambda_2 = 0 \\ y = 1 - x \\ x + y^2 - 1 \leq 0 \end{cases} \Rightarrow \begin{cases} x = -\lambda_2 \\ y = -1 - \lambda_2 \\ -1 - \lambda_2 = 1 + \lambda_2 \\ x + y^2 - 1 \leq 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda_2 = -1 < 0 \\ 1 + 0 - 1 \leq 0 \end{cases}$$

Since $\lambda_2 = -1 < 0$ the point $(1, 0)$ could be a minimum point.

4) caso $\lambda_1 \neq 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = y + 1 - \lambda_1 + \lambda_2 = 0 \\ \Lambda'_y = x - 2\lambda_1 y + \lambda_2 = 0 \\ y = 1 - x \\ x = 1 - y^2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \\ 1 - \lambda_1 + \lambda_2 = 0 \\ 1 + \lambda_2 = 0 \end{cases} \cup \begin{cases} x = 0 \\ y = 1 \\ 2 - \lambda_1 + \lambda_2 = 0 \\ -2\lambda_1 + \lambda_2 = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 1 \\ y = 0 \\ \lambda_1 = 0 \\ \lambda_2 = -1 < 0 \end{cases} \cup \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = -2 < 0 \\ \lambda_2 = -4 < 0 \end{cases}$$

Since the λ values are not positive, points $(1, 0)$ and $(0, 1)$ could be minimum points.

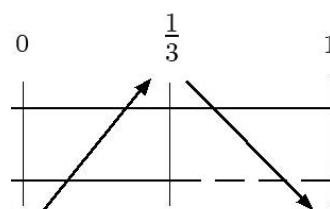
Let's study the objective function on the constraint $x = 1 - y^2$.

It is $f(1 - y^2, y) = (1 - y^2)(y + 1) = y + 1 - y^3 - y^2 \Rightarrow f'(y) = 1 - 3y^2 - 2y \geq 0$.

So $3y^2 + 2y - 1 \leq 0 \Rightarrow y = \frac{-1 \pm \sqrt{1+3}}{3} = \frac{-1 \pm 2}{3}$. Therefore:

$f'(y) \geq 0$ for $-1 \leq y \leq \frac{1}{3} \Rightarrow 0 \leq y \leq \frac{1}{3}$. If $y = \frac{1}{3} \Rightarrow x = \frac{8}{9}$.

At the point $\left(\frac{8}{9}, \frac{1}{3}\right)$ there is the Maximum with $f\left(\frac{8}{9}, \frac{1}{3}\right) = \frac{32}{27}$.

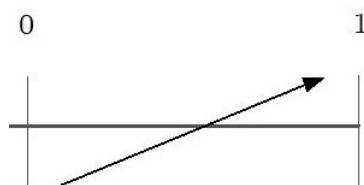


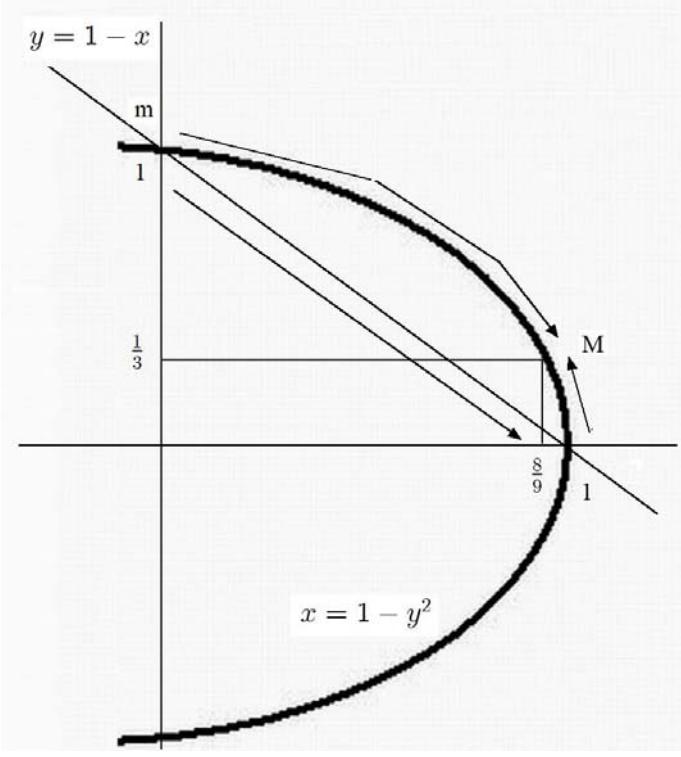
Let's study the objective function on the constraint $y = 1 - x$.

It is $f(x, 1 - x) = x(1 - x + 1) = 2x - x^2 \Rightarrow f'(x) = 2 - 2x \geq 0$ for $0 \leq x \leq 1$.

If $x = 0 \Rightarrow y = 1$. The point $(0, 1)$ is the minimum point with $f(0, 1) = 0$.

If $x = 1 \Rightarrow y = 0$. The point $(1, 0)$ it is neither a minimum nor a maximum point.





II M 3) Given the equation $f(x, y, z) = 2xy - e^{z-x} - e^{z-y} = 0$, satisfied at $(1, 1, 1)$, verify that an implicit function $(x, y) \rightarrow z(x, y)$ can be defined with it and then calculate the first order derivatives of such function.

The function $f(x, y, z)$ is differentiable $\forall (x, y, z) \in \mathbb{R}^3$, and $f(1, 1, 1) = 2 - 1 - 1 = 0$. It is: $\nabla f(x, y, z) = (2y + e^{z-x}; 2x + e^{z-y}; -e^{z-x} - e^{z-y})$.

So $\nabla f(1, 1, 1) = (3; 3; -2)$.

Since $f'_z(1, 1, 1) = -2 \neq 0$ it is possible to define an implicit function $(x, y) \rightarrow z(x, y)$ whose first order derivatives will be equal to:

$$\frac{\partial z}{\partial x}(1; 1) = -\frac{f'_x(1, 1, 1)}{f'_z(1, 1, 1)} = -\frac{3}{-2} = \frac{3}{2}; \quad \frac{\partial z}{\partial y}(1; 1) = -\frac{f'_y(1, 1, 1)}{f'_z(1, 1, 1)} = -\frac{3}{-2} = \frac{3}{2}.$$

II M 4) Given the function $f(x, y, z) = x^2 + y^2 + z^2 - x^2y - yz$, analyze the nature of its stationary points.

We determine the stationary points of the function. Applying the first order conditions we have:

$$\begin{aligned} \nabla f(x, y, z) = \emptyset &\Rightarrow \begin{cases} f'_x = 2x - 2xy = 2x(1-y) = 0 \\ f'_y = 2y - x^2 - z = 0 \\ f'_z = 2z - y = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \text{ and also:} \\ z = 0 \end{cases} \\ \begin{cases} x^2 = 2 - \frac{1}{2} = \frac{3}{2} \\ y = 1 \\ z = \frac{1}{2} \end{cases} &\Rightarrow \begin{cases} x = \sqrt{\frac{3}{2}} \\ y = 1 \text{ and} \\ z = \frac{1}{2} \end{cases} \quad \begin{cases} x = -\sqrt{\frac{3}{2}} \\ y = 1 \\ z = \frac{1}{2} \end{cases}. \end{aligned}$$

We have then found three stationary points: $(0, 0, 0)$, $\left(\sqrt{\frac{3}{2}}, 1, \frac{1}{2}\right)$ and $\left(-\sqrt{\frac{3}{2}}, 1, \frac{1}{2}\right)$.

We apply the second order conditions. From $\mathbb{H}(x, y, z) = \begin{vmatrix} 2 - 2y & -2x & 0 \\ -2x & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$ we get:

$$\mathbb{H}(0, 0, 0) = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} \Rightarrow \begin{cases} |\mathbb{H}_1| = 2 > 0 \\ |\mathbb{H}_2| = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} > 0; |\mathbb{H}_2| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} > 0 \\ |\mathbb{H}_3| = 2 \cdot \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} > 0 \end{cases};$$

since $\begin{cases} |\mathbb{H}_1| > 0 \\ |\mathbb{H}_2| > 0 \\ |\mathbb{H}_3| > 0 \end{cases}$ the point $(0, 0, 0)$ is a minimum point;

$$\mathbb{H}\left(\sqrt{\frac{3}{2}}, 1, \frac{1}{2}\right) = \begin{vmatrix} 0 & -2\sqrt{\frac{3}{2}} & 0 \\ -2\sqrt{\frac{3}{2}} & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} \Rightarrow |\mathbb{H}_2| = \begin{vmatrix} 0 & -2\sqrt{\frac{3}{2}} \\ -2\sqrt{\frac{3}{2}} & 2 \end{vmatrix} < 0;$$

since $|\mathbb{H}_2| < 0$ the point $\left(\sqrt{\frac{3}{2}}, 1, \frac{1}{2}\right)$ is a saddle point;

$$\mathbb{H}\left(-\sqrt{\frac{3}{2}}, 1, \frac{1}{2}\right) = \begin{vmatrix} 0 & 2\sqrt{\frac{3}{2}} & 0 \\ 2\sqrt{\frac{3}{2}} & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} \Rightarrow |\mathbb{H}_2| = \begin{vmatrix} 0 & 2\sqrt{\frac{3}{2}} \\ 2\sqrt{\frac{3}{2}} & 2 \end{vmatrix} < 0;$$

since $|\mathbb{H}_2| < 0$ the point $\left(-\sqrt{\frac{3}{2}}, 1, \frac{1}{2}\right)$ is a saddle point.