QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS
TASK 9/2/2021

I M 1) Two complex numbers are given z_1 and z_2 such that, written in trigonometric form, have moduli equal respectively to 4 and $\frac{1}{2}$, and arguments respectively equal to $\frac{17}{5}\pi$ and $\frac{23}{20}\pi$. Calculate their quotient $\frac{z_1}{z_2}$.

It is :

$$\frac{z_1}{z_2} = \frac{4\left(\cos\frac{17\pi}{5} + i\sin\frac{17\pi}{5}\right)}{\frac{1}{2}\left(\cos\frac{23\pi}{20} + i\sin\frac{23\pi}{20}\right)} = 8\left(\cos\left(\frac{17\pi}{5} - \frac{23\pi}{20}\right) + i\sin\left(\frac{17\pi}{5} - \frac{23\pi}{20}\right)\right) =$$

$$\frac{z_1}{z_2} = 8\left(\cos\frac{45\pi}{20} + i\sin\frac{45\pi}{20}\right) = 8\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right) = 8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) =$$

$$8\left(\frac{1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right) = 4\sqrt{2}\,(1+i)\,.$$

I M 2) Determine an orthogonal matrix that diagonalizes $\mathbb{A} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{Vmatrix}$.

The matrix is a symmetric one and therefore certainly diagonalizable by means of an orthogonal matrix.

From $|\mathbb{A} - \lambda\,\mathbb{I}| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$ we get:

$(1-\lambda)\big((1-\lambda)^2 - 1\big) = (1-\lambda)\big(\lambda^2 - 2\lambda\big) = \lambda\,(1-\lambda)(\lambda-2) = 0\,.$

So the eigenvalues $\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2\,.$

To find an eigenvector relative to $\lambda_1 = 0$ we solve the system:

$$\begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} x = 0 \\ y - z = 0 \\ -y + z = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = z \end{cases}$$ and so the eigenvector:

$(0,1,1)$ and its unit vector $\left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

To find an eigenvector relative to $\lambda_1 = 1$ we solve the system:

$$\begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{Vmatrix} \cdot \begin{Vmatrix} x \\ y \\ z \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \\ 0 \end{Vmatrix} \Rightarrow \begin{cases} 0 = 0 \\ z = 0 \\ y = 0 \end{cases} \Rightarrow \begin{cases} \forall\, x \\ y = 0 \\ z = 0 \end{cases}$$ and so the eigenvector:

$(1,0,0)$ which is an unit vector.

To find an eigenvector relative to $\lambda_1 = 2$ we solve the system:

$$\left\|\begin{matrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\| = \left\|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right\| \Rightarrow \begin{cases} x = 0 \\ y + z = 0 \\ -y - z = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -z \end{cases}$$ and so the eigen-vector: $(0, 1, -1)$ and its unit vector $\left(0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

So the orthogonal matrix is $\mathbb{U} = \left\|\begin{matrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{matrix}\right\|$.

I M 3) Given the matrix $\mathbb{A} = \left\|\begin{matrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{matrix}\right\|$, after having determined the dimensions of the Kernel and the Image of the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbb{X}) = \mathbb{A} \cdot \mathbb{X}$, find all the vectors $\mathbb{X}$ that in this linear map coincide with their image: $f(\mathbb{X}) = \mathbb{X}$.

To solve the problem firstly we calculate the rank of the matrix $\mathbb{A} = \left\|\begin{matrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{matrix}\right\|$.

Since $|\mathbb{A}| = \left|\begin{matrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{matrix}\right| = 2 \cdot \left|\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}\right| = 0$ but $\left|\begin{matrix} 1 & 0 \\ 0 & 2 \end{matrix}\right| = 2 \neq 0 \Rightarrow \text{Rank}(\mathbb{A}) = 2$.

So $\text{Dim}(\text{Imm}) = 2$ and $\text{Dim}(\text{Ker}) = 3 - 2 = 1$.

To solve the equality $f(\mathbb{X}) = \mathbb{X}$ we have to solve the system:

$$\left\|\begin{matrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{matrix}\right\| \cdot \left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\| = \left\|\begin{matrix} x \\ y \\ z \end{matrix}\right\| \Rightarrow \begin{cases} x + z = x \\ 2y = y \\ x + z = z \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}.$$

Only the null vector satisfies the given equation.

I M 4) Given the matrix $\mathbb{A} = \left\|\begin{matrix} 1 & -2 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 5 & -4 & 2 & 1 \end{matrix}\right\|$ and the linear map $f : \mathbb{R}^4 \to \mathbb{R}^3$, $f(\mathbb{X}) = \mathbb{A} \cdot \mathbb{X}$, determine if the vector $\mathbb{Y} = (1, 1, 4)$ belongs to the Image of that linear map.

To check if a vector belongs to the Image of a linear map corresponds perfectly to seeing if the linear system having that matrix as its coefficient matrix admits solutions when the vector of the known terms is the given vector. So we use the Rouchè-Capelli Theorem and we calculate the rank of the matrix and the rank of the augmented matrix .

$$(\mathbb{A}|\mathbb{Y}) = \left\|\begin{matrix} 1 & -2 & 0 & 1 & | & 1 \\ 1 & 1 & 1 & -1 & | & 1 \\ 5 & -4 & 2 & 1 & | & 4 \end{matrix}\right\|.$$

By elementary operations on the rows $(R_2 \leftarrow R_2 - R_1)$, $(R_3 \leftarrow R_3 - 5R_1)$, we get :

$$\left\|\begin{matrix} 1 & -2 & 0 & 1 & | & 1 \\ 1 & 1 & 1 & -1 & | & 1 \\ 5 & -4 & 2 & 1 & | & 4 \end{matrix}\right\| \to \left\|\begin{matrix} 1 & -2 & 0 & 1 & | & 1 \\ 0 & 3 & 1 & -2 & | & 0 \\ 0 & 6 & 2 & -4 & | & -1 \end{matrix}\right\|.$$

By elementary operations on the rows $(R_3 \leftarrow R_3 - 2R_2)$, we get :

$$\left\|\begin{matrix}1 & -2 & 0 & 1 & | & 1\\ 0 & 3 & 1 & -2 & | & 0\\ 0 & 6 & 2 & -4 & | & -1\end{matrix}\right\| \to \left\|\begin{matrix}1 & -2 & 0 & 1 & | & 1\\ 0 & 3 & 1 & -2 & | & 0\\ 0 & 0 & 0 & 0 & | & -1\end{matrix}\right\|.$$

From this it immediately follows that $\text{Rank}(\mathbb{A}) = 2 < 3 = \text{Rank}(\mathbb{A}|\mathbb{Y})$, therefore the system has no solutions and the vector $\mathbb{Y} = (1,1,4)$ does not belong to the Image of the linear map.

II M 1) Given the equation $f(x,y) = e^{x^2-y} - e^{y^2-x} = 0$, verify that at the point $\mathrm{P}_0 = (1,1)$ the hypotheses of Dini's Theorem are satisfied and then find the equation of the tangent line to the implicit function $y = y(x)$ so determined at $x_0 = 1$.

The function $f(x,y) = e^{x^2-y} - e^{y^2-x}$ is differentiable $\forall\,(x,y) \in \mathbb{R}^2$.
From $\nabla f(x,y) = \left(2x\,e^{x^2-y} + e^{y^2-x};\ -e^{x^2-y} - 2y\,e^{y^2-x}\right)$ we get $\nabla f(1,1) = (3;\,-3)$.
Since $f'_y(1,1) = -3 \neq 0$ it is possible to define an implicit function $y = y(x)$ whose first order derivative is equal to: $\dfrac{\mathrm{d}y}{\mathrm{d}x}(1) = -\dfrac{f'_x(1,1)}{f'_y(1,1)} = -\dfrac{3}{-3} = 1$.
So the equation of the tangent line to the function $y = y(x)$ at $x_0 = 1$ is $y - 1 = 1\,(x-1)$ that is $y = x$.

II M 2) Solve the problem $\begin{cases}\text{Max/min } f(x,y) = 2x^2 + y^2\\ \text{u.c.}: x^2 + 2y^2 \le 1\end{cases}$.

The objective function of the problem is a continuous function, the constraint defines a feasible region (ellipse) which is a closed and bounded (compact) set, and so we can apply Weierstrass Theorem. Surely the function admits maximum value and minimum value.
To solve the problem we use the Kuhn-Tucker conditions.
We write the problem as $\begin{cases}\text{Max/min } f(x,y) = 2x^2 + y^2\\ \text{s.v.}: x^2 + 2y^2 - 1 \le 0\end{cases}$.
We form the Lagrangian function:
$\Lambda(x,y,\lambda) = 2x^2 + y^2 - \lambda\left(x^2 + 2y^2 - 1\right)$.

By applying the first order conditions we have:

1) case $\lambda = 0$:
$\begin{cases}\Lambda'_x = 4x = 0\\ \Lambda'_y = 2y = 0\\ x^2 + 2y^2 \le 1\end{cases} \Rightarrow \begin{cases}x = 0\\ y = 0\\ 0 + 0 \le 1\end{cases}$; $\mathbb{H}(0,0) = \left\|\begin{matrix}4 & 0\\ 0 & 2\end{matrix}\right\|$ and so $(0,0)$ is a minimum point internal to the feasible region .

2) case $\lambda \neq 0$:
$\begin{cases}\Lambda'_x = 4x - 2\lambda x = 2x\,(2-\lambda) = 0\\ \Lambda'_y = 2y - 4\lambda y = 2y\,(1-2\lambda) = 0\\ x^2 + 2y^2 = 1\end{cases} \Rightarrow \begin{cases}x = 0\\ y = 0\\ 0 + 0 = 1 \text{ impossible}\end{cases} \cup \begin{cases}\lambda = 2\\ \lambda = \frac{1}{2}\\ \text{impossible}\end{cases}$ or

$\begin{cases}\Lambda'_x = 2x\,(2-\lambda) = 0\\ \Lambda'_y = 2y\,(1-2\lambda) = 0\\ x^2 + 2y^2 = 1\end{cases} \Rightarrow \begin{cases}x = 0\\ \lambda = \frac{1}{2}\\ y^2 = \frac{1}{2}\end{cases} \cup \begin{cases}y = 0\\ \lambda = 2\\ x^2 = 1\end{cases}$ hence the four solutions:

$$\begin{cases} x=0 \\ y=\frac{1}{\sqrt{2}} \\ \lambda=\frac{1}{2} \end{cases} \cup \begin{cases} x=0 \\ y=-\frac{1}{\sqrt{2}} \\ \lambda=\frac{1}{2} \end{cases} \cup \begin{cases} x=1 \\ y=0 \\ \lambda=2 \end{cases} \cup \begin{cases} x=-1 \\ y=0 \\ \lambda=2 \end{cases}.$$

Since the values of λ are positive, these points could be maximum points.

Given the presence of only one constraint, we can check the nature of these points using the bordered Hessian matrix.

It is $\overline{\mathbb{H}}(x,y,\lambda)=\left\|\begin{matrix} 0 & 2x & 4y \\ 2x & 4-2\lambda & 0 \\ 4y & 0 & 2-4\lambda \end{matrix}\right\|$. So:

$\left|\overline{\mathbb{H}}\left(0,\frac{1}{\sqrt{2}},\frac{1}{2}\right)\right|=\begin{vmatrix} 0 & 0 & 2\sqrt{2} \\ 0 & 3 & 0 \\ 2\sqrt{2} & 0 & 0 \end{vmatrix}=3\cdot\begin{vmatrix} 0 & 2\sqrt{2} \\ 2\sqrt{2} & 0 \end{vmatrix}=3(-8)<0$ therefore the point would be reported as a minimum point while the multiplier indicates it as a maximum point . So $\left(0,\frac{1}{\sqrt{2}},\frac{1}{2}\right)$ it is neither a maximum nor a minimum point.

$\left|\overline{\mathbb{H}}\left(0,-\frac{1}{\sqrt{2}},\frac{1}{2}\right)\right|=\begin{vmatrix} 0 & 0 & -2\sqrt{2} \\ 0 & 3 & 0 \\ -2\sqrt{2} & 0 & 0 \end{vmatrix}=3\cdot\begin{vmatrix} 0 & -2\sqrt{2} \\ -2\sqrt{2} & 0 \end{vmatrix}=3(-8)<0$ therefore the point would be reported as a minimum point while the multiplier indicates it as a maximum point. So $\left(0,-\frac{1}{\sqrt{2}},\frac{1}{2}\right)$ it is neither a maximum nor a minimum point.

$\left|\overline{\mathbb{H}}(1,0,2)\right|=\begin{vmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & -6 \end{vmatrix}=(-6)\cdot\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}=(-6)(-4)>0$ then the point is reported as a maximum point as reported by the multiplier. So $(1,0,2)$ is a maximum point.

$\left|\overline{\mathbb{H}}(-1,0,2)\right|=\begin{vmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -6 \end{vmatrix}=(-6)\cdot\begin{vmatrix} 0 & -2 \\ -2 & 0 \end{vmatrix}=(-6)(-4)>0$ then the point is reported as a maximum point as reported by the multiplier. So $(-1,0,2)$ is a maximum point.

In conclusion, $(0,0)$ is the minimum point with $f(0,0)=0$ while $(1,0)$ and $(-1,0)$ are the maximum points with $f(1,0)=f(-1,0)=2$.

II M 3) Given the function $f(x,y)=x^2+y^2+x\,y$ find all directions $v=(\cos\alpha,\sin\alpha)$ for which it results $\mathcal{D}_v f(1,1)=0$.

The function $f(x,y)=x^2+y^2+x\,y$ is differentiable $\forall\,(x,y)\in\mathbb{R}^2$.
So $\mathcal{D}_v f(\mathrm{P}_0)=\nabla f(\mathrm{P}_0)\cdot v$. From $\nabla f(x,y)=(2x+y;2y+x)$ we get :
$\nabla f(1,1)=(3;3)$ and so $\mathcal{D}_v f(1,1)=(3;3)\cdot(\cos\alpha,\sin\alpha)=3\cos\alpha+3\sin\alpha=0$
from which we get $\cos\alpha=-\sin\alpha$ valid for $\alpha=\frac{3\pi}{4}$ and for $\alpha=\frac{7\pi}{4}$.

II M 4) Given the function $f(x,y,z)=x^2+y^2+z^2-x-xy^2$, determine the nature of its stationary points.

We determine the stationary points of the function. Applying the first order conditions $\nabla f(x,y,z)=\mathbb{O}$ we have:

$$\begin{cases} f'_x = 2x-1-y^2=0 \\ f'_y = 2y-2xy=2y\,(1-x)=0 \\ f'_z = 2z=0 \end{cases} \Rightarrow \begin{cases} 2x-1-y^2=0 \\ y=0 \\ z=0 \end{cases} \cup \begin{cases} 2x-1-y^2=0 \\ x=1 \\ z=0 \end{cases}.$$

From the first system:

$$\Rightarrow \begin{cases} 2x-1-y^2=0 \\ y=0 \\ z=0 \end{cases} \Rightarrow \begin{cases} 2x-1=0 \\ y=0 \\ z=0 \end{cases} \Rightarrow \begin{cases} x=\frac{1}{2} \\ y=0 \\ z=0 \end{cases};$$

From the second system:

$$\begin{cases} 2x-1-y^2=0 \\ x=1 \\ z=0 \end{cases} \Rightarrow \begin{cases} y^2=1 \\ x=1 \\ z=0 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=1 \\ z=0 \end{cases} \cup \begin{cases} x=1 \\ y=-1 \\ z=0 \end{cases}.$$

We have then found three stationary points: $\left(\frac{1}{2},0,0\right)$, $(1,1,0)$ and $(1,-1,0)$.

We apply the second order conditions. From $\mathbb{H}(x,y,z)=\left\|\begin{matrix} 2 & -2y & 0 \\ -2y & 2-2x & 0 \\ 0 & 0 & 2 \end{matrix}\right\|$ we get:

$$\mathbb{H}\left(\frac{1}{2},0,0\right)=\left\|\begin{matrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{matrix}\right\| \Rightarrow \begin{cases} |\mathbb{H}_1|=2>0; |\mathbb{H}_1|=1>0 \\ |\mathbb{H}_2|=\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}>0;\ |\mathbb{H}_2|=\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}>0 \\ |\mathbb{H}_3|=2\cdot\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}>0 \end{cases};$$

since $\begin{cases} |\mathbb{H}_1|>0 \\ |\mathbb{H}_2|>0 \\ |\mathbb{H}_3|>0 \end{cases}$ the point $\left(\frac{1}{2},0,0\right)$ is a minimum point;

$$\mathbb{H}(1,1,0)=\left\|\begin{matrix} 2 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{matrix}\right\| \Rightarrow |\mathbb{H}_2|=\begin{vmatrix} 2 & -2 \\ -2 & 0 \end{vmatrix}<0;$$

since $|\mathbb{H}_2|<0$ the point $(1,1,0)$ is a saddle point;

$$\mathbb{H}(1,-1,0)=\left\|\begin{matrix} 2 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{matrix}\right\| \Rightarrow |\mathbb{H}_2|=\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix}<0;$$

since $|\mathbb{H}_2|<0$ the point $(1,-1,0)$ is a saddle point.