QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS
TASK 15/3/2021

I M 1) If $z = e\,(\cos\alpha + i\sin\alpha)$, since $e^{1-3i}\cdot z = e^{2+i}$, determine α.

$e^{1-3i}\cdot z = e^{2+i} \Rightarrow z = \dfrac{e^{2+i}}{e^{1-3i}} = e^{2+i-(1-3i)} = e^{1+4i} = e\,(\cos 4 + i\sin 4)\,.$

Therefore $\alpha = 4\,.$

I M 2) In the basis for $\mathbb{R}^3$ formed by the vectors $\mathbb{V}_1 = (1,2,-1)$, $\mathbb{V}_2 = (2,1,1)$ and $\mathbb{V}_3 = (x_1,x_2,x_3)$, the coordinates of the vector $\mathbb{Y} = (1,3,0)$ are $(2,-2,1)$. Determine $\mathbb{X}_3$.

The problem is equivalent to solving the following linear system:

$$\left\|\begin{matrix} 1 & 2 & x_1 \\ 2 & 1 & x_2 \\ -1 & 1 & x_3 \end{matrix}\right\| \cdot \left\|\begin{matrix} 2 \\ -2 \\ 1 \end{matrix}\right\| = \left\|\begin{matrix} 1 \\ 3 \\ 0 \end{matrix}\right\| \Rightarrow \begin{cases} 2-4+x_1 = 1 \\ 4-2+x_2 = 3 \\ -2-2+x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = 3 \\ x_2 = 1 \\ x_3 = 4 \end{cases}.$$

And so $\mathbb{V}_3 = (5,3,0)\,.$

I M 3) Given the matrix $\mathbb{A} = \left\|\begin{matrix} 2 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & m & 4 \end{matrix}\right\|$, determine, on varying the parameter m, the existence of multiple eigenvalues and determine if, for these values, the matrix is diagonalizable.

Let's determine the eigenvalues of the matrix:

From $|\mathbb{A} - \lambda\,\mathbb{I}| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 2 & 1-\lambda & 2 \\ 1 & m & 4-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ \lambda-3 & m & 4-\lambda \end{vmatrix} = 0$ we get:

$(1-\lambda)[(1-\lambda)(4-\lambda)-2m] + (\lambda-3)(0-(1-\lambda)) =$

$= (1-\lambda)\big(\lambda^2 - 5\lambda + 4 - 2m - \lambda + 3\big) = (1-\lambda)\big(\lambda^2 - 6\lambda - 2m + 7\big) = 0\,.$

There are two ways to obtain multiple eigenvalues :

$-$ The polynomial $\big(\lambda^2 - 6\lambda - 2m + 7\big)$ vanishes for $\lambda = 1$;

$-$ The polynomial $\big(\lambda^2 - 6\lambda - 2m + 7\big)$ admits two identical roots.

First case. For $\lambda = 1$ we get: $1 - 6 - 2m + 7 = 0 \Rightarrow 2m = 2 \Rightarrow m = 1$ and so:

$\lambda^2 - 6\lambda + 5 = 0 \Rightarrow \lambda = 3\pm\sqrt{9-5} = 3\pm 2$ and so we get $\lambda = 1$ and $\lambda = 5\,.$

For $m = 1$ and $\lambda = 1$ we get:

$\|\mathbb{A} - 1\cdot\mathbb{I}\| = \left\|\begin{matrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 1 & 1 & 3 \end{matrix}\right\| \Rightarrow \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \neq 0$ and so $\text{Rank}\,(\|\mathbb{A} - 1\cdot\mathbb{I}\|) = 2$ and therefore $m_1^g = 3 - 2 = 1 < m_1^a = 2$ and si the matrix is not diagonalizable.

Second case. $\lambda^2 - 6\lambda - 2m + 7 = 0 \Rightarrow \lambda = 3\pm\sqrt{9+2m-7} = 3\pm\sqrt{2+2m}\,.$

$2 + 2m = 0$ for $m = -1$ and the eigenvalues are $\lambda_1 = 1, \lambda_2 = \lambda_3 = 3$.

For $m = -1$ and $\lambda = 3$ we get:

$$\|\mathbb{A}-3\cdot\mathbb{I}\| = \begin{Vmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{Vmatrix} \Rightarrow \begin{vmatrix} -1 & 0 \\ 2 & -2 \end{vmatrix} = 2 \neq 0 \text{ and so } \operatorname{Rank}(\|\mathbb{A}-3\cdot\mathbb{I}\|) = 2$$

and therefore $m_3^g = 3-2 = 1 < m_3^a = 2$ and so the matrix is not diagonalizable.
For any other value of m we will have only simple eigenvalues and therefore the matrix will be diagonalizable.

I M 4) Given the linear system $\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 2 \\ 2x_1 + x_2 - x_3 + 2x_4 = 1 \\ x_1 - 4x_2 + 7x_3 + x_4 = k \end{cases}$, find the values of the parameter k for which the system has solutions.

To check if the linear system admits solutions we use the Rouchè-Capelli Theorem and we calculate the rank of the matrix and the rank of the augmented matrix :

$$(\mathbb{A}|\mathbb{Y}) = \begin{Vmatrix} 1 & -1 & 2 & 1 & | & 2 \\ 2 & 1 & -1 & 2 & | & 1 \\ 1 & -4 & 7 & 1 & | & k \end{Vmatrix}.$$

By elementary operations on the rows $(R_2 \leftarrow R_2 - 2R_1)$, $(R_3 \leftarrow R_3 - R_1)$, we get :

$$\begin{Vmatrix} 1 & -1 & 2 & 1 & | & 2 \\ 2 & 1 & -1 & 2 & | & 1 \\ 1 & -4 & 7 & 1 & | & k \end{Vmatrix} \to \begin{Vmatrix} 1 & -1 & 2 & 1 & | & 2 \\ 0 & 3 & -5 & 0 & | & -3 \\ 0 & -3 & 5 & 0 & | & k-2 \end{Vmatrix}.$$

By elementary operations on the rows $(R_3 \leftarrow R_3 + R_2)$, we get :

$$\begin{Vmatrix} 1 & -1 & 2 & 1 & | & 2 \\ 0 & 3 & -5 & 0 & | & -3 \\ 0 & -3 & 5 & 0 & | & k-2 \end{Vmatrix} \to \begin{Vmatrix} 1 & -1 & 2 & 1 & | & 2 \\ 0 & 3 & -5 & 0 & | & -3 \\ 0 & 0 & 0 & 0 & | & k-5 \end{Vmatrix}.$$

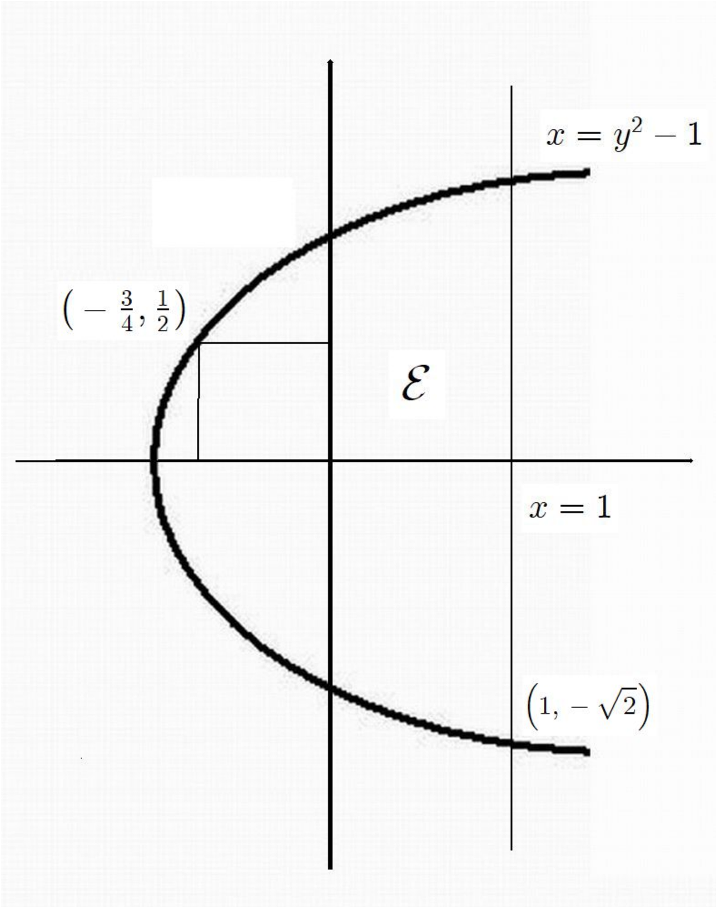
From this it follows that $\operatorname{Rank}(\mathbb{A}) = 2 = \operatorname{Rank}(\mathbb{A}|\mathbb{Y})$ only for $k=5$, and for this value the system has $\infty^{4-2} = \infty^2$ solutions. For $k \neq 5$ it is $\operatorname{Rank}(\mathbb{A}) = 2 < \operatorname{Rank}(\mathbb{A}|\mathbb{Y}) = 3$ and the system has no solutions.

II M 1) Given the equation $f(x,y) = x^3 - 3xy + y^3 - 3 = 0$ and the point $P = (-1,1)$ that satisfies such equation, find the equation of the tangent line to the graphic of the implicit function $y = y(x)$ defined at the point $x = -1$.

The function $f(x,y) = x^3 - 3xy + y^3 - 3$ is differentiable $\forall (x,y) \in \mathbb{R}^2$.
From $\nabla f(x,y) = (3x^2 - 3y; 3y^2 - 3x)$ we get $\nabla f(-1,1) = (0;6)$.
Since $f_y'(-1,1) = 6 \neq 0$ it is possible to define an implicit function $y = y(x)$ whose first order derivative is equal to: $\dfrac{\mathrm{d}y}{\mathrm{d}x}(-1) = -\dfrac{f_x'(-1,1)}{f_y'(-1,1)} = -\dfrac{0}{6} = 0$. So the equation of the tangent line to the function $y = y(x)$ at $x_0 = -1$ is $y - 1 = 0\cdot(x+1)$ that is $y = 1$.

II M 2) Solve the problem $\begin{cases} \text{Max/min } f(x,y) = x - y \\ \text{u.c.: } y^2 - 1 \le x \le 1 \end{cases}$.

The objective function of the problem is a continuous function, the constraint defines a feasible region which is a closed and bounded (compact) set, and so we can apply Weierstrass Theorem. Surely the function admits maximum value and minimum value.



To solve the problem we use the Kuhn-Tucker conditions.

We write the problem as $\begin{cases} \text{Max/min } f(x,y) = x - y \\ \text{u.c.: } \begin{cases} y^2 - x - 1 \le 0 \\ x - 1 \le 0 \end{cases} \end{cases}$.

We form the Lagrangian function:

$\Lambda(x, y, \lambda) = x - y - \lambda_1\left(y^2 - x - 1\right) - \lambda_2(x - 1)$.

By applying the first order conditions we have:

1) case $\lambda_1 = 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = 1 \ne 0 \\ \Lambda'_y = -1 \ne 0 \\ y^2 - x - 1 \le 0 \\ x - 1 \le 0 \end{cases} \Rightarrow \text{no solutions}.$$

2) case $\lambda_1 \ne 0, \lambda_2 = 0$:

$$\begin{cases} \Lambda'_x = 1 + \lambda_1 = 0 \\ \Lambda'_y = -1 - 2\lambda_1\, y = 0 \\ x = y^2 - 1 \\ x \le 1 \end{cases} \Rightarrow \begin{cases} \lambda_1 = -1 < 0 \\ y = \frac{1}{2} \\ x = \frac{1}{4} - 1 = -\frac{3}{4} \\ -\frac{3}{4} \le 1 \end{cases};$$

Since $\lambda_1 = -1 < 0$ the point $\left(-\dfrac{3}{4}, \dfrac{1}{2}\right)$ could be a minimum point.

3) case $\lambda_1 = 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = 1 - \lambda_2 = 0 \\ \Lambda'_y = -1 \neq 0 \\ x = 1 \\ y^2 - x - 1 \leq 0 \end{cases} \Rightarrow \text{no solutions} .$$

4) case $\lambda_1 \neq 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = 1 + \lambda_1 - \lambda_2 = 0 \\ \Lambda'_y = -1 - 2\lambda_1 \, y = 0 \\ x = 1 \\ x = y^2 - 1 \end{cases} \Rightarrow \begin{cases} \Lambda'_x = 1 + \lambda_1 - \lambda_2 = 0 \\ \Lambda'_y = -1 - 2\lambda_1 \, y = 0 \\ x = 1 \\ y^2 = 2 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = 1 \\ y = \sqrt{2} \\ 1 + \lambda_1 - \lambda_2 = 0 \\ -1 - 2\lambda_1 \, y = 0 \end{cases} \cup \begin{cases} x = 1 \\ y = -\sqrt{2} \\ 1 + \lambda_1 - \lambda_2 = 0 \\ -1 - 2\lambda_1 \, y = 0 \end{cases} .$$

$$\begin{cases} x = 1 \\ y = \sqrt{2} \\ 1 + \lambda_1 - \lambda_2 = 0 \\ -1 - 2\lambda_1 \, y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = \sqrt{2} \\ 1 + \lambda_1 - \lambda_2 = 0 \\ -1 - 2\sqrt{2}\,\lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = \sqrt{2} \\ \lambda_1 = -\frac{1}{2\sqrt{2}} < 0 \\ \lambda_2 = 1 + \lambda_1 = \frac{2\sqrt{2}-1}{2\sqrt{2}} > 0 \end{cases} .$$

Since $\lambda_1 = -\dfrac{1}{2\sqrt{2}} < 0$ and $\lambda_2 = \dfrac{2\sqrt{2}-1}{2\sqrt{2}} > 0$ the point $\left(1, \sqrt{2}\right)$ it is neither a maximum nor a minimum point .

$$\begin{cases} x = 1 \\ y = -\sqrt{2} \\ 1 + \lambda_1 - \lambda_2 = 0 \\ -1 - 2\lambda_1 \, y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -\sqrt{2} \\ 1 + \lambda_1 - \lambda_2 = 0 \\ -1 + 2\sqrt{2}\,\lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = -\sqrt{2} \\ \lambda_1 = \frac{1}{2\sqrt{2}} > 0 \\ \lambda_2 = 1 + \lambda_1 = \frac{2\sqrt{2}+1}{2\sqrt{2}} > 0 \end{cases} .$$

Since $\lambda_1 = \dfrac{1}{2\sqrt{2}} > 0$ and $\lambda_2 = \dfrac{2\sqrt{2}+1}{2\sqrt{2}} > 0$ the point $\left(1, -\sqrt{2}\right)$ could be a maximum point.

However, having found only two solutions, for the Weierstrass Theorem, the point $\left(-\dfrac{3}{4}, \dfrac{1}{2}\right)$ is the minimum point, with $f\left(-\dfrac{3}{4}, \dfrac{1}{2}\right) = -\dfrac{5}{4}$ while the point $\left(1, -\sqrt{2}\right)$ is the maximum point with $f\left(1, -\sqrt{2}\right) = 1 + \sqrt{2}$.

II M 3) Given $f(x, y) = xy - 2x + y$, let v and w be the unit vectors of $(1, 1)$ and $(1, -1)$; determine if there are points P where $\mathcal{D}_v f(P) = \sqrt{2}$ and $\mathcal{D}_w f(P) = -\sqrt{2}$.

The function $f(x, y) = xy - 2x + y$ is differentiable $\forall\, (x, y) \in \mathbb{R}^2$.
So $\mathcal{D}_v f(P) = \nabla f(P) \cdot v$ and $\mathcal{D}_w f(P) = \nabla f(P) \cdot w$.
From $\nabla f(x, y) = (y - 2; x + 1)$, $v = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$, $w = \left(\dfrac{1}{\sqrt{2}}, -\dfrac{1}{\sqrt{2}}\right)$ we get :

$$\begin{cases} (y-2; x+1) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \sqrt{2} \\ (y-2; x+1) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = -\sqrt{2} \end{cases} \Rightarrow \begin{cases} x+y-1=2 \\ y-x-3=-2 \end{cases} \Rightarrow \begin{cases} x+y=3 \\ x-y=-1 \end{cases}$$

$$\Rightarrow \begin{cases} y-1+y=3 \\ x=y-1 \end{cases} \Rightarrow \begin{cases} 2y=4 \\ x=y-1 \end{cases} \Rightarrow \begin{cases} x=1 \\ y=2 \end{cases}.$$ So $P=(1,2)$.

II M 4) Find the pair (x,y) for which the determinant of the matrix $\mathbb{A} = \begin{Vmatrix} xy & y \\ x & x+y \end{Vmatrix}$ has a minimum value.

From $|\mathbb{A}| = \begin{vmatrix} xy & y \\ x & x+y \end{vmatrix} = xy(x+y) - xy$ we get $f(x,y) = x^2y + xy^2 - xy$.

We determine the stationary points of the function. Applying the first order conditions $\nabla f(x,y) = \mathbb{O}$ we have:

$$\begin{cases} f'_x = 2xy + y^2 - y = 0 \\ f'_y = x^2 + 2xy - x = 0 \end{cases} \Rightarrow \begin{cases} y(2x+y-1) = 0 \\ x(x+2y-1) = 0 \end{cases}.$$ So four possible cases:

$$\begin{cases} y=0 \\ x=0 \end{cases} \cup \begin{cases} y=0 \\ x+2y-1=0 \end{cases} \cup \begin{cases} 2x+y-1=0 \\ x=0 \end{cases} \cup \begin{cases} 2x+y-1=0 \\ x+2y-1=0 \end{cases}$$ and so:

$$\begin{cases} x=0 \\ y=0 \end{cases} \cup \begin{cases} x=1 \\ y=0 \end{cases} \cup \begin{cases} x=0 \\ y=1 \end{cases} \cup \begin{cases} y=1-2x \\ x+2-4x-1=0 \end{cases} \Rightarrow \begin{cases} x=\frac{1}{3} \\ y=\frac{1}{3} \end{cases}.$$

From $\mathbb{H}(x,y) = \begin{Vmatrix} 2y & 2x+2y-1 \\ 2x+2y-1 & 2x \end{Vmatrix}$ we get:

$\mathbb{H}(0,0) = \begin{Vmatrix} 0 & -1 \\ -1 & 0 \end{Vmatrix}$: since $|\mathbb{H}_2| = -1 < 0$ the point $(0,0)$ is a saddle point;

$\mathbb{H}(1,0) = \begin{Vmatrix} 0 & 1 \\ 1 & 2 \end{Vmatrix}$: since $|\mathbb{H}_2| = -1 < 0$ the point $(1,0)$ is a saddle point;

$\mathbb{H}(0,1) = \begin{Vmatrix} 2 & 1 \\ 1 & 0 \end{Vmatrix}$: since $|\mathbb{H}_2| = -1 < 0$ the point $(1,0)$ is a saddle point;

$\mathbb{H}\left(\frac{1}{3}, \frac{1}{3}\right) = \begin{Vmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{Vmatrix}$: since $\begin{cases} |\mathbb{H}_1| = \frac{2}{3} > 0 \\ |\mathbb{H}_2| = \frac{4}{9} - \frac{1}{9} = \frac{1}{3} > 0 \end{cases}$ the point $\left(\frac{1}{3}, \frac{1}{3}\right)$ is a minimum point, the solution of the problem.