QUANTITATIVE METHODS for ECONOMIC APPLICATIONS
MATHEMATICS for ECONOMIC APPLICATIONS
TASK 31/5/2021

I M 1) Find z if $z^3 = 1 - i$.

$$1 - i = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) = \sqrt{2}\left(\cos\frac{7}{4}\pi + i\sin\frac{7}{4}\pi\right).$$

From $z^3 = 1 - i$ it follows $z = \sqrt[3]{1-i}$ and so:

$$z = \sqrt[6]{2}\left(\cos\left(\frac{7}{12}\pi + k\,\frac{2\pi}{3}\right) + i\sin\left(\frac{7}{12}\pi + k\,\frac{2\pi}{3}\right)\right), 0 \le k \le 2.$$

For $k = 0 : z = \sqrt[6]{2}\left(\cos\frac{7}{12}\pi + i\sin\frac{7}{12}\pi\right)$;

for $k = 1 : z = \sqrt[6]{2}\left(\cos\frac{15}{12}\pi + i\sin\frac{15}{12}\pi\right) = \sqrt[6]{2}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right)$;

for $k = 2 : z = \sqrt[6]{2}\left(\cos\frac{23}{12}\pi + i\sin\frac{23}{12}\pi\right)$.

I M 2) Given the matrix $\mathbb{A} = \begin{Vmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 1 & -3 & 2 \end{Vmatrix}$ and linear map $\mathbb{R}^3 \to \mathbb{R}^3 : \mathbb{A}\cdot\mathbb{X} = \mathbb{Y}$, calculate the dimensions of its Kernel and its Image. Then given the two linearly independent vectors $\mathbb{X}_1 = (2,1,0)$ and $\mathbb{X}_2 = (1,2,2)$ determine if $\mathbb{Y}_1 = \mathbb{A}\cdot\mathbb{X}_1$ and $\mathbb{Y}_2 = \mathbb{A}\cdot\mathbb{X}_2$ are also linearly independent.

To check the dimensions of the Kernel and the Image we need to calculate the rank of the matrix. By elementary operations on the rows $(R_2 \leftarrow R_2 - 2R_1)$, $(R_3 \leftarrow R_3 - R_1)$, and then $(R_3 \leftarrow R_3 - R_2)$ we get :

$$\begin{Vmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 1 & -3 & 2 \end{Vmatrix} \to \begin{Vmatrix} 1 & 3 & -1 \\ 0 & -6 & 3 \\ 0 & -6 & 3 \end{Vmatrix} \to \begin{Vmatrix} 1 & 3 & -1 \\ 0 & -6 & 3 \\ 0 & 0 & 0 \end{Vmatrix}.$$

Since $\begin{vmatrix} 1 & 3 \\ 0 & -6 \end{vmatrix} = -6 \ne 0 \Rightarrow \text{Rank}(\mathbb{A}) = 2 \Rightarrow \text{Dim(Imm)} = 2, \text{Dim(Ker)} = 3 - 2 = 1$.

$$\mathbb{Y}_1 = \mathbb{A}\cdot\mathbb{X}_1 = \begin{Vmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 1 & -3 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} 2 \\ 1 \\ 0 \end{Vmatrix} = \begin{Vmatrix} 2+3+0 \\ 4+0+0 \\ 2-3+0 \end{Vmatrix} = \begin{Vmatrix} 5 \\ 4 \\ -1 \end{Vmatrix};$$

$$\mathbb{Y}_2 = \mathbb{A}\cdot\mathbb{X}_2 = \begin{Vmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 1 & -3 & 2 \end{Vmatrix} \cdot \begin{Vmatrix} 1 \\ 2 \\ 2 \end{Vmatrix} = \begin{Vmatrix} 1+6-2 \\ 2+0+2 \\ 1-6+4 \end{Vmatrix} = \begin{Vmatrix} 5 \\ 4 \\ -1 \end{Vmatrix}.$$

So $\mathbb{Y}_1 = \mathbb{A}\cdot\mathbb{X}_1 = \mathbb{Y}_2 = \mathbb{A}\cdot\mathbb{X}_2$ and $\mathbb{Y}_1$ and $\mathbb{Y}_2$ are linearly dependent vectors.

I M 3) Given the linear system $\begin{cases} 2x + 3y - z = 1 \\ 2x + y + 2z = 2 \\ 2x + 5y + kz = m \end{cases}$, determine for which values of the parameters k and m it admits none, only one or infinite solutions.

To check when the linear system admits none, only one or infinite solutions we use the Rouchè-Capelli Theorem and we calculate the rank of the matrix and the rank of the augmented matrix :

$(\mathbb{A}|\mathbb{Y}) = \left\| \begin{matrix} 2 & 3 & -1 & | & 1 \\ 2 & 1 & 2 & | & 2 \\ 2 & 5 & k & | & m \end{matrix} \right\|$. By elementary operations on the rows:

$(R_2 \leftarrow R_2 - R_1)$, $(R_3 \leftarrow R_3 - R_1)$, and then $(R_3 \leftarrow R_3 + R_2)$ we get :

$$\left\| \begin{matrix} 2 & 3 & -1 & | & 1 \\ 2 & 1 & 2 & | & 2 \\ 2 & 5 & k & | & m \end{matrix} \right\| \to \left\| \begin{matrix} 2 & 3 & -1 & | & 1 \\ 0 & -2 & 3 & | & 1 \\ 0 & 2 & k+1 & | & m-1 \end{matrix} \right\| \to \left\| \begin{matrix} 2 & 3 & -1 & | & 1 \\ 0 & -2 & 3 & | & 1 \\ 0 & 0 & k+4 & | & m \end{matrix} \right\|$$

and so we see that:

$-$ if $k \neq -4 \Rightarrow \text{Rank}(\mathbb{A}) = 3 = \text{Rank}(\mathbb{A}|\mathbb{Y})$: the system has one and only one solution;

$-$ if $k = -4$ and $m = 0 \Rightarrow \text{Rank}(\mathbb{A}) = 2 = \text{Rank}(\mathbb{A}|\mathbb{Y})$: the system has ∞^1 solutions;

$-$ if $k = -4$ and $m \neq 0 \Rightarrow \text{Rank}(\mathbb{A}) = 2 < \text{Rank}(\mathbb{A}|\mathbb{Y}) = 3$: the system has no solutions.

I M 4) Given the matrix $\mathbb{A} = \left\| \begin{matrix} 3 & 2 & 1 \\ 1 & 4 & 4 \\ 1 & 2 & 5 \end{matrix} \right\|$, knowing that it admits the eigenvalue $\lambda = 2$, determine if it is diagonalizable and if it admits orthogonal eigenvectors.

Let's determine the eigenvalues of the matrix:

From $|\mathbb{A} - \lambda\,\mathbb{I}| = \left| \begin{matrix} 3-\lambda & 2 & 1 \\ 1 & 4-\lambda & 4 \\ 1 & 2 & 5-\lambda \end{matrix} \right| = \left| \begin{matrix} 3-\lambda & 2 & 1 \\ 0 & 2-\lambda & \lambda-1 \\ 1 & 2 & 5-\lambda \end{matrix} \right| = 0$ we get:

$= (3-\lambda)[(2-\lambda)(5-\lambda) - 2\,(\lambda-1)] + 1\,[2\,(\lambda-1) - (2-\lambda)] =$

$= (3-\lambda)\left(\lambda^2 - 9\lambda + 12\right) + (3\lambda - 4) = -\lambda^3 + 12\lambda^2 - 36\lambda + 32 = 0$.

From $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$, for $\lambda = 2$ we get $8 - 48 + 72 - 32 = 0$.

By Ruffini's rule we get:

	1	−12	36	− 32
2		2	−20	32
	1	−10	16	0

So $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = (\lambda - 2)\left(\lambda^2 - 10\lambda + 16\right) = 0$ and

$\lambda^2 - 10\lambda + 16 = 0 \Rightarrow \lambda = 5 \pm \sqrt{25 - 16} = 5 \pm \sqrt{9} = 5 \pm 3 \Rightarrow \lambda = 2$ and $\lambda = 8$.

So the eigenvalues are $\lambda_1 = \lambda_2 = 2, \lambda_3 = 8$.

To check the diagonalizability of the matrix we must study only the multiple eigenvalue and so, for $\lambda = 2$ we get: $\|\mathbb{A} - 2 \cdot \mathbb{I}\| = \left\| \begin{matrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ 1 & 2 & 3 \end{matrix} \right\| \Rightarrow \left| \begin{matrix} 2 & 1 \\ 2 & 4 \end{matrix} \right| = 6 \neq 0$.

So $\text{Rank}\,(\|\mathbb{A} - 2 \cdot \mathbb{I}\|) = 2 \Rightarrow m_2^g = 3 - 2 = 1 < 2 = m_2^a$ and the matrix is not diagonalizable. The dimension of the eigenspace corresponding to $\lambda = 2$ is equal to 1.

To get the eigenvectors corresponding to $\lambda = 2$ we must solve the homogeneous system :

$$\|\mathbb{A}-2\cdot\mathbb{I}\|\cdot\mathbb{X}=\mathbb{O}\Rightarrow\begin{Vmatrix}1&2&1\\1&2&4\\1&2&3\end{Vmatrix}\cdot\begin{Vmatrix}x\\y\\z\end{Vmatrix}=\begin{Vmatrix}0\\0\\0\end{Vmatrix}\Rightarrow\begin{Vmatrix}1&2&1\\0&0&3\\0&0&2\end{Vmatrix}\cdot\begin{Vmatrix}x\\y\\z\end{Vmatrix}=\begin{Vmatrix}0\\0\\0\end{Vmatrix}\Rightarrow$$

$$\Rightarrow\begin{cases}x+2y+z=0\\3z=0\\2z=0\end{cases}\Rightarrow\begin{cases}x+2y=0\\z=0\end{cases}\Rightarrow\begin{cases}x=-2y\\\forall y\\z=0\end{cases}\Rightarrow(-2k,k,0),\ k\in\mathbb{R}$$ the eigenvectors corresponding to $\lambda=2$.

To get the eigenvectors corresponding to $\lambda=8$ we must solve the homogeneous system :

$$\|\mathbb{A}-8\cdot\mathbb{I}\|\cdot\mathbb{X}=\mathbb{O}\Rightarrow\begin{Vmatrix}-5&2&1\\1&-4&4\\1&2&-3\end{Vmatrix}\cdot\begin{Vmatrix}x\\y\\z\end{Vmatrix}=\begin{Vmatrix}0\\0\\0\end{Vmatrix}\Rightarrow$$

$$\Rightarrow\begin{Vmatrix}0&12&-14\\0&-6&7\\1&2&-3\end{Vmatrix}\cdot\begin{Vmatrix}x\\y\\z\end{Vmatrix}=\begin{Vmatrix}0\\0\\0\end{Vmatrix}\Rightarrow\begin{cases}x+2y-3z=0\\-6y+7z=0\\6y-7z=0\end{cases}\Rightarrow\begin{cases}x+2y-\frac{18}{7}y=0\\z=\frac{6}{7}y\end{cases}\Rightarrow$$

$$\Rightarrow\begin{cases}x=-\frac{4}{7}y\\\forall y\\z=\frac{6}{7}y\end{cases}\Rightarrow\left(-\frac{4}{7}k,k,\frac{6}{7}k\right),\ k\in\mathbb{R}$$ the eigenvectors corresponding to $\lambda=8$.

For $k=1$ we get the two eigenvectors $\mathbb{X}_2=(-2,1,0)$ and $\mathbb{X}_8=\left(-\frac{4}{7},1,\frac{6}{7}\right)$ and since $\mathbb{X}_2\cdot\mathbb{X}_8=(-2,1,0)\cdot\left(-\frac{4}{7},1,\frac{6}{7}\right)=\frac{8}{7}+1\neq0$ the two eigenvectors are not orthogonal.

II M 1) Given the equation $f(x,y)=e^{(x-1)\,y}-y^2\,e^{x-1}=0$, find all the points $(1,y)$ that satisfy this equation, verify in these points the applicability of the Dini's Theorem to obtain an implicit function $y=y(x)$, and calculate the first derivative of this function at the points found.

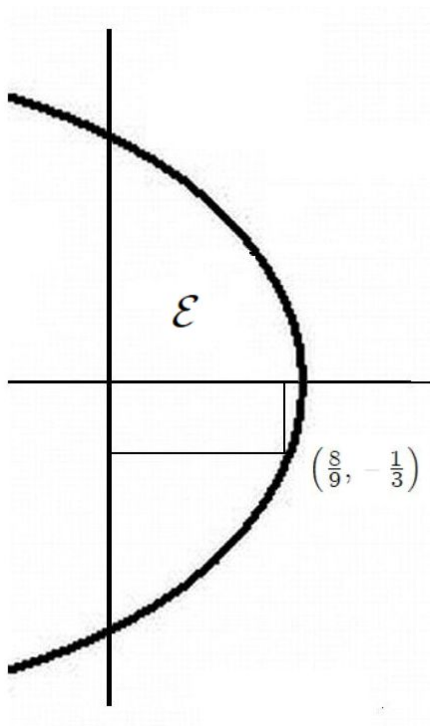
The function $f(x,y)=e^{(x-1)\,y}-y^2\,e^{x-1}=0$ is differentiable $\forall\,(x,y)\in\mathbb{R}^2$.
Substituting $(1,y)$ in the equation we get $e^{(1-1)\,y}-y^2\,e^{1-1}=1-y^2=0\Rightarrow y=\pm1$.
We have therefore got the two points $(1,1)$ and $(1,-1)$. From:
$\nabla f(x,y)=\left(y\,e^{(x-1)\,y}-y^2\,e^{x-1};(x-1)e^{(x-1)\,y}-2y\,e^{x-1}\right)$ we get $\nabla f(1,1)=(0;-2)$ and $\nabla f(1,-1)=(-2;2)$. Since $f'_y(1,\pm1)\neq0$ it is possible to define implicit functions $y=y(x)$ whose first order derivatives are equal to:
$\frac{\mathrm{d}y}{\mathrm{d}x}(1)=-\frac{f'_x(1,1)}{f'_y(1,1)}=-\frac{0}{-2}=0$ and $\frac{\mathrm{d}y}{\mathrm{d}x}(1)=-\frac{f'_x(1,-1)}{f'_y(1,-1)}=-\frac{-2}{2}=1$.

II M 2) Solve the problem $\begin{cases}\text{Max/min } f(x,y)=x\,(y-1)\\\text{u.c.:}\begin{cases}y^2+x-1\leq0\\x\geq0\end{cases}\end{cases}$. It may be useful to study the sign of the function.

The objective function of the problem is a continuous function, the constraints define a feasible region which is a closed and bounded (compact) set, and so we can apply Weierstrass Theorem. Surely the function admits maximum value and minimum value.
If we study the sign of the function we get:

$$f(x,y)=x\,(y-1)>0\Rightarrow\begin{cases}x>0\\y-1>0\end{cases}\Rightarrow\begin{cases}x>0\\y>1\end{cases}\text{ or }\begin{cases}x<0\\y-1<0\end{cases}\Rightarrow\begin{cases}x<0\\y<1\end{cases}.$$

So in the feasible region the objective function is always negative except in the points $x=0$ in which it is equal to zero. So all the points $x=0$ of the feasible region that belong to the y axis are maximum points.



To complete the solution of the problem we use the Kuhn-Tucker conditions.

We write the problem as $\begin{cases} \text{Max/min } f(x,y)=x\,(y-1) \\ \text{u.c.: } \begin{cases} y^2+x-1\le 0 \\ -x\le 0 \end{cases} \end{cases}$.

We form the Lagrangian function:

$\Lambda(x,y,\lambda)=xy-x-\lambda_1\left(y^2+x-1\right)-\lambda_2(-x)$.

By applying the first order conditions we have:

1) case $\lambda_1=0, \lambda_2=0$:

$$\begin{cases} \Lambda'_x=y-1=0 \\ \Lambda'_y=x=0 \\ y^2+x-1\le 0 \\ x\ge 0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \\ 1+0-1\le 0 \\ 0\ge 0 \end{cases}.$$ We already know the nature of the points $x=0$.

2) case $\lambda_1\neq 0, \lambda_2=0$:

$$\begin{cases} \Lambda'_x=y-1-\lambda_1=0 \\ \Lambda'_y=x-2\lambda_1\,y=0 \\ x=1-y^2 \\ x\ge 0 \end{cases} \Rightarrow \begin{cases} y=1+\lambda_1 \\ x=2\lambda_1(1+\lambda_1)=2\lambda_1+2\lambda_1^2 \\ 2\lambda_1+2\lambda_1^2=1-1-\lambda_1^2-2\lambda_1 \\ x\ge 0 \end{cases} \Rightarrow \begin{cases} y=1+\lambda_1 \\ x=2\lambda_1+2\lambda_1^2 \\ 3\lambda_1^2+4\lambda_1=0 \\ x\ge 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = 2\lambda_1 + 2\lambda_1^2 \\ y = 1 + \lambda_1 \\ \lambda_1(3\lambda_1 + 4) = 0 \\ x \geq 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = 0 \\ x \geq 0 \end{cases} \cup \begin{cases} x = 2\lambda_1\, y = 2\left(-\frac{4}{3}\right)\left(-\frac{1}{3}\right) = \frac{8}{9} \\ y = 1 - \frac{4}{3} = -\frac{1}{3} \\ \lambda_1 = -\frac{4}{3} \\ x \geq 0 \end{cases} ;$$

we already know the nature of the point $(0,1)$ while $\left(\frac{8}{9}, -\frac{1}{3}\right)$, since $\lambda_1 = -\dfrac{4}{3} < 0$ may be a minimum point.

3) case $\lambda_1 = 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = y - 1 + \lambda_2 = 0 \\ \Lambda'_y = x = 0 \\ x = 0 \\ y^2 + x - 1 \leq 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ x = 0 \\ \lambda_2 = 1 - y \\ y^2 + x - 1 \leq 0 \end{cases}$$

: we know the nature of the points $x = 0$.

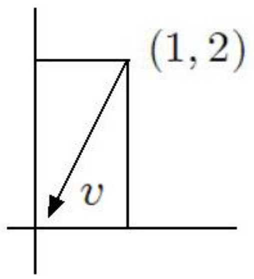
4) case $\lambda_1 \neq 0, \lambda_2 \neq 0$:

$$\begin{cases} \Lambda'_x = y - 1 - \lambda_1 + \lambda_2 = 0 \\ \Lambda'_y = x - 2\lambda_1\, y = 0 \\ x = 0 \\ x = 1 - y^2 \end{cases} \Rightarrow \begin{cases} y - 1 - \lambda_1 + \lambda_2 = 0 \\ x - 2\lambda_1\, y = 0 \\ x = 0 \\ y^2 = 1 \end{cases} \Rightarrow$$

$$\begin{cases} x = 0 \\ y = 1 \\ \lambda_1 - \lambda_2 = 0 \\ \lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 1 \\ \lambda_1 = 0 \\ \lambda_2 = 0 \end{cases} \cup \begin{cases} x = 0 \\ y = -1 \\ \lambda_1 - \lambda_2 = -2 \\ \lambda_1 = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \\ \lambda_1 = 0 \\ \lambda_2 = 2 \end{cases} .$$

We already know that the points $x = 0$ are maximum points, with $f(0, y) = 0$ while $\left(\frac{8}{9}, -\frac{1}{3}\right)$ is the minimum point, with $f\left(\dfrac{8}{9}, -\dfrac{1}{3}\right) = -\dfrac{32}{27}$.

II M 3) Given $f(x,y) = x\,y^2 - x^2$, calculate $\mathcal{D}_v f(P_0)$, where v represents the direction from $P_0 = (1,2)$ to the origin $(0,0)$.



The function $f(x,y) = x\,y^2 - x^2$ is a differentiable function and so $\mathcal{D}_v f(P_0) = \nabla f(P_0) \cdot v$. From $\nabla f(x,y) = \left(y^2 - 2x, 2xy\right)$ we get $\nabla f(1,2) = (2,4)$ and since $\|(1,2)\| = \sqrt{5}$ we get $v = \left(-\dfrac{1}{\sqrt{5}}, -\dfrac{2}{\sqrt{5}}\right)$ and finally $\mathcal{D}_v f(1,2) = (2,4) \cdot \left(-\dfrac{1}{\sqrt{5}}, -\dfrac{2}{\sqrt{5}}\right) = -\dfrac{10}{\sqrt{5}}$.

II M 4) Determine the value of x that minimizes the value of the determinant of the matrix $\mathbb{A} = \left\| \begin{matrix} 2x-1 & 2 & x-5 \\ 2x & 1 & x+1 \\ x+1 & 3 & x+2 \end{matrix} \right\|$.

By elementary operations on the rows: $(R_1 \leftarrow R_1 - 2R_2)$, $(R_3 \leftarrow R_3 - 3R_2)$ we get:

$$|\mathbb{A}| = \begin{vmatrix} 2x-1 & 2 & x-5 \\ 2x & 1 & x+1 \\ x+1 & 3 & x+2 \end{vmatrix} = \begin{vmatrix} 2x-1-4x & 2-2 & x-5-2(x+1) \\ 2x & 1 & x+1 \\ x+1-6x & 3-3 & x+2-3(x+1) \end{vmatrix} =$$

$$= \begin{vmatrix} -6x-1 & 0 & -x-7 \\ 2x & 1 & x+1 \\ -5x+1 & 0 & -2x-1 \end{vmatrix} = (-6x-1)(-2x-1) - (-x-7)(-5x+1) =$$

$$= 12x^2 + 6x + 2x + 1 - \left(5x^2 - x + 35x - 7\right) = 7x^2 - 26x + 8 = \mathbb{D}(x).$$

From $\mathbb{D}'(x) = 14x - 26 \geq 0$ for $x \geq \dfrac{26}{14} = \dfrac{13}{7}$ the value of x that minimizes the value of the determinant is $x = \dfrac{13}{7}$.